6/11 PP Talk Thank orgs.

Bruhat graphs and pattern avoidance
w/ Chris Conklin

will mention problem + sol'n first,
then motivation - goal is motivation,
problem + sol'n are initial observations

Outline
1 ~~Problem~~ Definition and problem
2 Answers
3 Motivation.

Bruhat graph $\mathcal{B}(w)$ for a permutation: w.
vertices labelled by perm.

e.g. 2431 321

remove inversions to create new vertices. Edges connect vertices differing by transposition.

remove inversions
arrows up.
always bipartite

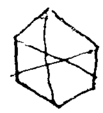
Show 3412

Observe: If w contains v, then $\mathcal{B}(w)$ contains $\mathcal{B}(v)$ as a subgraph (in fact embedded at the top vertex). The converse is false. (132 + 231, or 321 and 3412)

Problems: Which Bruhat graphs can be drawn in the plane? On a torus?

2 Answers Def: $\ell(w) = \#\text{inv}(w)$.
(length)

Thm: $\mathcal{B}(w)$ is planar if and only if
1) w avoids 321
2) $\ell(w) < 4$

Pf: If w contains 321,  is part of graph - not planar.
If w contains 3412, same
~~If $\ell(w) \geq 4$, either w contains 321 or 3412 or not.~~
If ~~$\mathcal{B}$~~ w avoids 321 + 3412, $\mathcal{B}(w)$ is the edge graph of a cube (Tenner, 2006) of dim. $\ell(w)$. This is planar iff $\ell(w) < 4$.

~~Yhm~~

~~Restatements - 29 patterns~~

$\ell(w) < 4$ means containing 21 at most 3 times. This condition can be converted to <u>finitely</u> many patterns (Atkinson 1999)

In this case, 29 patterns.

For a torus:

<u>Thm</u>: $\mathcal{B}(w)$ can be drawn on a torus if

a) w avoids 3412

b) $\ell(w) < 5$

c) If $\ell(w) = 4$, w avoids 321.

<u>Pf</u>: - 3412 (Eldredge) - $K_{3,3}$ + cube.

- contain 321 + $\ell(w) = 4$ ($\ell(w) = 3$ same graph as 321)

- has $K_{3,3} \times$ —.

- no 321 or 3412: cube.

Can't find computation. (57?)

3 Motivation

Some properties of permutations are characterized by avoiding infinitely many patterns - e.g. sortability by deques (Pratt '73)
But:

Thm (-, Billey-Weed, '09)

$P_{id,w}(1) \leq 2$ if and only if w avoids 66 patterns

Kazhdan-Lusztig polynomial at $q=1$ ($\frac{1}{2}$ hr to define, a semester to explain)
Pos int coeff. - only known pf requires very sophisticated alg. geom.

Thm (Billey-Braden '04)
$P_{id,w}(1) \leq k$ is always characterized by avoiding a list of patterns.

Question (Billey '09) Is this list always finite? (Speculation - for $k=3$, seems likely; several hundred)

Thm (Delanoy '06): $P_{id,w}(q)$ depends only on $\mathcal{B}(w)$. (not proven for $P_{v,w}(q)$ in general)

We see ~~larger~~ finite sets of patterns for other Schubert phenomena:

~~smooth (2)~~
~~(c)~~ ~~(6)~~
~~(d)~~

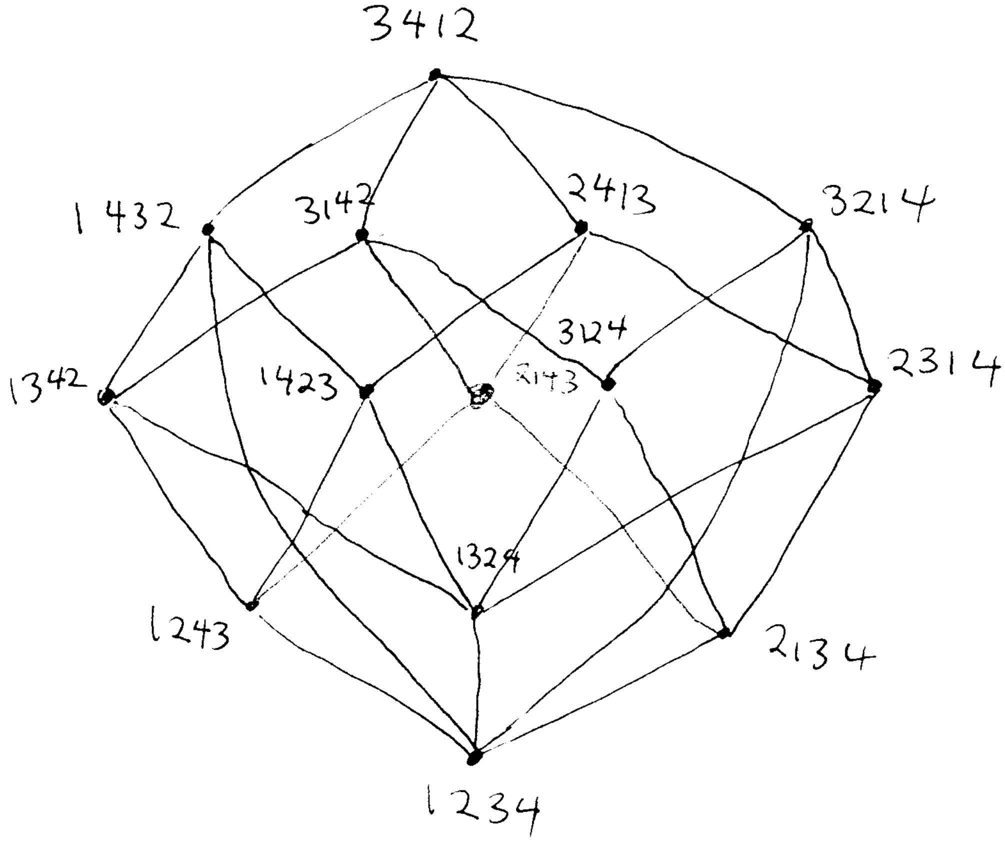
Whenever the answer involves classical patterns, the number seems finite.

General question: If a property of permutations is

1) Characterized by classical patterns
2) Depends only on the Bruhat graph,

is it always characterized by a finite list of patterns?

Why planar + torus graphs? these are beginnings of graph minor theory.


3412
1432
3142
2413
3214
3124
1342
1423
2143
2314
1324
1243
2134
1234

C