PERMUTATION PATTERNS 2012

Small permutation classes

Vincent Vatter
University of Florida

→

Built using deck.js and MathJax.
Hit m for a menu, or the arrow keys to navigate.

PERMUTATION PATTERNS 2012

Small permutation classes

Vincent Vatter
University of Florida

Based on:

- Vatter. Small permutation classes. *Proc. Lond. Math. Soc. (3)* 103 (2011), 879–921.
- Albert, Atkinson, and Vatter. Subclasses of the separable permutations. *Bull. Lond. Math. Soc.* 43 (2011), 859–870.
- Albert, Atkinson, Bouvel, Ruškuc, and Vatter. Geometric grid classes of permutations. *Trans. Amer. Math. Soc.*, to appear.
- Albert, Ruškuc, and Vatter. Inflations of geometric grid classes of permutations. arXiv:1202.1833v1 [math.CO].

Built using deck.js and MathJax.
Hit m for a menu, or the arrow keys to navigate.

← →

Background

This is classical pattern containment, so we write $\sigma \leq \pi$ if π contains a subsequence in the same relative order as σ.

A permutation class is a downset in this order.

- $\mathcal{C}_n$ denotes the permutations in the class $\mathcal{C}$ of length n.
- The basis of the class $\mathcal{C}$ is the minimal permutations *not* in $\mathcal{C}$;
$$\mathrm{Av}(B) = \{\pi : \pi \text{ avoids } \beta \text{ for all } \beta \in B\}.$$
- The generating function of the class $\mathcal{C}$ is
$$\sum_{n\in\mathbb{N}} |\mathcal{C}_n| x^n = \sum_{\pi\in\mathcal{C}} x^{|\pi|}.$$
- The (upper) growth rate of the class $\mathcal{C}$ is defined as
$$\mathrm{gr}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

Background

This is classical pattern containment, so we write $\sigma \leq \pi$ if π contains a subsequence in the same relative order as σ.

A permutation class is a downset in this order.

- $\mathcal{C}_n$ denotes the permutations in the class $\mathcal{C}$ of length n.
- The basis of the class $\mathcal{C}$ is the minimal permutations *not* in $\mathcal{C}$;
$$\mathrm{Av}(B) = \{\pi : \pi \text{ avoids } \beta \text{ for all } \beta \in B\}.$$
- The generating function of the class $\mathcal{C}$ is
$$\sum_{n\in\mathbb{N}} |\mathcal{C}_n| x^n = \sum_{\pi\in\mathcal{C}} x^{|\pi|}.$$
- The (upper) growth rate of the class $\mathcal{C}$ is defined as
$$\mathrm{gr}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

Does $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$ always exist?

Background

Conjecture (Noonan and Zeilberger 1996): For any finite basis B, the class $\mathrm{Av}(B)$ has a D-finite generating function.

(f is D-finite if it and all of its derivatives span a finite dimensional vector space over $\mathbb{C}[[x]]$.)

Background

Conjecture (Noonan and Zeilberger 1996): For any finite basis B, the class $\mathrm{Av}(B)$ has a D-finite generating function.

(f is D-finite if it and all of its derivatives span a finite dimensional vector space over $\mathbb{C}[[x]]$.)

- Conjecture (Zeilberger PP2005): The Noonan-Zeilberger Conjecture is false.

Background

Conjecture (Noonan and Zeilberger 1996): For any finite basis B, the class $\mathrm{Av}(B)$ has a D-finite generating function.

(f is D-finite if it and all of its derivatives span a finite dimensional vector space over $\mathbb{C}[[x]]$.)

- Conjecture (Zeilberger PP2005): The Noonan-Zeilberger Conjecture is false.
- Conjecture? (Zeilberger PP2005): "Not even God knows $\mathrm{Av}_{1000}(1324)$."

← →

Background

Conjecture (Noonan and Zeilberger 1996): For any finite basis B, the class $\mathrm{Av}(B)$ has a D-finite generating function.

(f is D-finite if it and all of its derivatives span a finite dimensional vector space over $\mathbb{C}[[x]]$.)

- Conjecture (Zeilberger PP2005): The Noonan-Zeilberger Conjecture is false.
- Conjecture? (Zeilberger PP2005): "Not even God knows $\mathrm{Av}_{1000}(1324)$."

Conjecture (Balogh, Bollobás, and Morris 2005): Growth rates are always algebraic integers.

Background

Conjecture (Noonan and Zeilberger 1996): For any finite basis B, the class $\mathrm{Av}(B)$ has a D-finite generating function.

(f is D-finite if it and all of its derivatives span a finite dimensional vector space over $\mathbb{C}[[x]]$.)

- Conjecture (Zeilberger PP2005): The Noonan-Zeilberger Conjecture is false.
- Conjecture? (Zeilberger PP2005): "Not even God knows $\mathrm{Av}_{1000}(1324)$."

Conjecture (Balogh, Bollobás, and Morris 2005): Growth rates are always algebraic integers.

- Theorem (Albert and Linton 2009): The set of growth rates contains an uncountable perfect set...

Growth rates

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

Growth rates

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

There is a phase transition at κ:

Growth rates

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

← There is a phase transition at κ: →

- κ is the first accumulation point of accumulation points of growth rates.

Growth rates

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

There is a phase transition at κ:

- κ is the first accumulation point of accumulation points of growth rates.
- κ is the first growth rate that admits an infinite antichain.

Growth rates

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

There is a phase transition at κ:

- κ is the first accumulation point of accumulation points of growth rates.
- κ is the first growth rate that admits an infinite antichain.
- Only countably many permutation classes have growth rates under κ.

Growth rates

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

There is a phase transition at κ:

- κ is the first accumulation point of accumulation points of growth rates.
- κ is the first growth rate that admits an infinite antichain.
- Only countably many permutation classes have growth rates under κ.
- All permutation classes of growth rate less than κ have rational generating functions (ARV 2012+).

Pre-background

ELSEVIER

Discrete Mathematics 195 (1999) 27–38

DISCRETE MATHEMATICS

Restricted permutations

M.D. Atkinson*

School of Mathematical and Computational Sciences, North Haugh, St Andrews, Fife KY16 9SS, UK

Received 6 November 1997; revised 11 March 1998; accepted 13 April 1998

Pre-background

ELSEVIER

Discrete Mathematics 195 (1999) 27–38

DISCRETE MATHEMATICS

Restricted permutations

M.D. Atkinson*

School of Mathematical and Computational Sciences, North Haugh, St Andrews, Fife KY16 9SS, UK

Received 6 November 1997; revised 11 March 1998; accepted 13 April 1998

- Initiated the *systematic* study of permutation classes.

Pre-background

ELSEVIER

Discrete Mathematics 195 (1999) 27–38

DISCRETE MATHEMATICS

Restricted permutations

M.D. Atkinson*

School of Mathematical and Computational Sciences, North Haugh, St Andrews, Fife KY16 9SS, UK

Received 6 November 1997; revised 11 March 1998; accepted 13 April 1998

- Initiated the *systematic* study of permutation classes.
- 59 citations on Google Scholar.

Pre-background

Order **19:** 101–113, 2002.

101

Partially Well-Ordered Closed Sets of Permutations

M. D. ATKINSON
Department of Computer Science, University of Otago, New Zealand.
E-mail: mike@cs.otago.ac.nz

M. M. MURPHY and N. RUŠKUC
School of Mathematics and Statistics, University of St Andrews, U.K.
E-mail: {max,nik}@mcs.st-and.ac.uk

(Received: 30 May 2001; accepted: 24 January 2002)

- Initiated the systematic study of infinite antichains of permutations.
- 44 citations on Google Scholar.

Pre-background

Order **19:** 101–113, 2002.

101

Partially Well-Ordered Closed Sets of Permutations

M. D. ATKINSON
Department of Computer Science, University of Otago, New Zealand.
E-mail: mike@cs.otago.ac.nz

M. M. MURPHY and N. RUŠKUC
School of Mathematics and Statistics, University of St Andrews, U.K.
E-mail: {max,nik}@mcs.st-and.ac.uk

(Received: 30 May 2001; accepted: 24 January 2002)

- Initiated the systematic study of infinite antichains of permutations.
- 44 citations on Google Scholar.
- Introduced $\mathcal{W}$-classes; precursors of monotone grid classes.

Pre-background

ELSEVIER

Discrete Mathematics 259 (2002) 19–36

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Restricted permutations and the wreath product

M.D. Atkinson[a,*], T. Stitt[b]

[a]*Department of Computer Science, University of Otago, P.O. Box 56, Dunedin, New Zealand*
[b]*School of Computer Science, North Haugh, St Andrews, Fife KY16 9SS, UK*

Received 17 August 1999; received in revised form 27 July 2001; accepted 28 January 2002

- Initiated the study of the substitution decomposition of permutations.
- 44 citations on Google Scholar.

Pre-background

ELSEVIER

Available at www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT•

Theoretical Computer Science 306 (2003) 85–100

Theoretical Computer Science

www.elsevier.com/locate/tcs

Regular closed sets of permutations

M.H. Albert[a], M.D. Atkinson[a,*], N. Ruškuc[b]

[a] *Department of Computer Science, University of Otago, New Zealand*
[b] *School of Mathematics and Statistics, University of St Andrews, UK*

Received 12 June 2002; received in revised form 19 February 2003; accepted 26 February 2003
Communicated by W. Szpankowski

- Initiated the use of formal language theory in the study of permutation classes.
- Established the first *general criterion* to show that permutation classes have rational generating functions.
- 29 citations on Google Scholar.

Growth rates review

- The jump from 1 to ϕ is the Fibonacci Dichotomy of Kaiser and Klazar (2003).
- Below $\kappa \approx 2.21$, we have a characterization of all growth rates (V 2011).
- Above $\lambda \approx 2.48$, all real numbers are growth rates (V 2010).

There is a phase transition at κ:

- κ is the first accumulation point of accumulation points of growth rates.
- κ is the first growth rate that admits an infinite antichain.
- Only countably many permutation classes have growth rates under κ.
- All permutation classes of growth rate less than κ have rational generating functions (ARV 2012+).

Outline of (rest of) talk

Tools:

- The substitution decomposition: blowing permutations up.
- Grid classes: chopping permutations up.

Small permutation classes:

- Structure.
- Enumeration.

Other recent (& future?) uses of these tools.

Substitution

Let π be a permutation of length m and $\alpha_1, \ldots, \alpha_m$ arbitrary permutations.

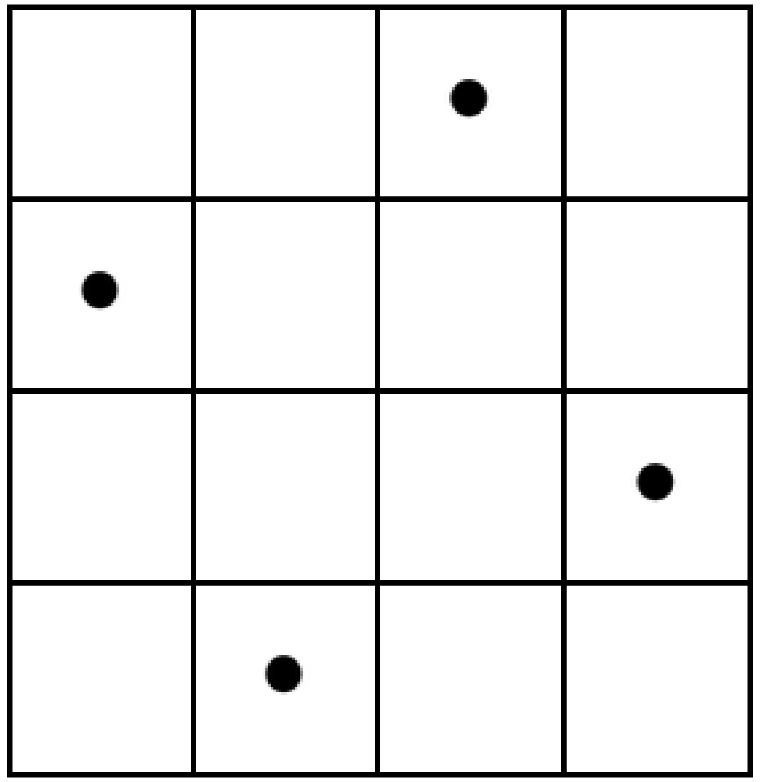
We form the inflation $\pi[\alpha_1, \ldots, \alpha_m]$ by replacing each entry $\pi(i)$ by an "interval" which is order isomorphic to α_i in such a way that the intervals themselves are order isomorphic to π.

Substitution

Let π be a permutation of length m and $\alpha_1, \ldots, \alpha_m$ arbitrary permutations.

We form the inflation $\pi[\alpha_1, \ldots, \alpha_m]$ by replacing each entry $\pi(i)$ by an "interval" which is order isomorphic to α_i in such a way that the intervals themselves are order isomorphic to π.

Example: 3142

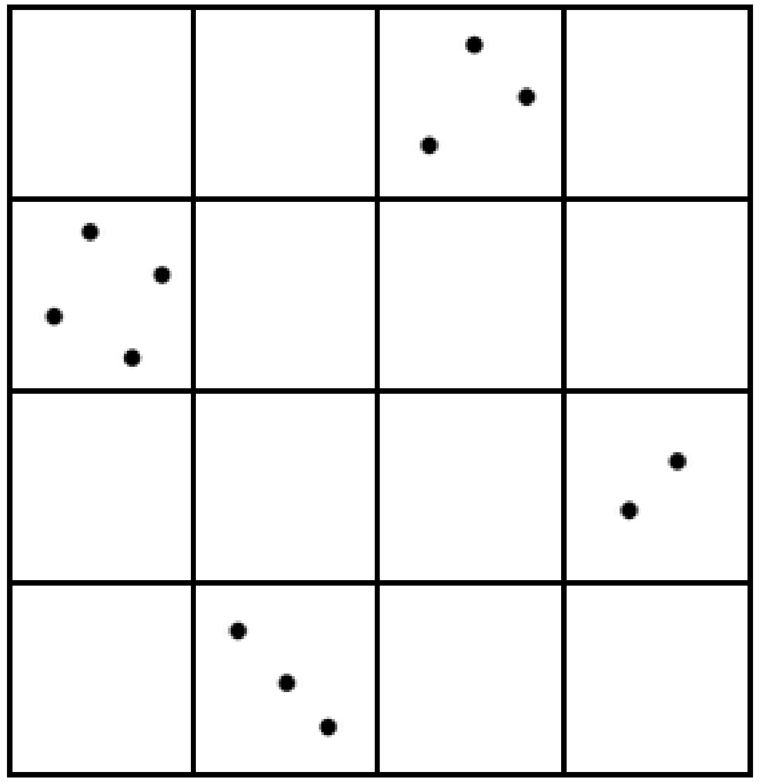


Substitution

Let π be a permutation of length m and $\alpha_1, \ldots, \alpha_m$ arbitrary permutations.

We form the inflation $\pi[\alpha_1, \ldots, \alpha_m]$ by replacing each entry $\pi(i)$ by an "interval" which is order isomorphic to α_i in such a way that the intervals themselves are order isomorphic to π.

Example: $3142[2413, 321, 132, 12]$



Substitution

Given permutation classes $\mathcal{C}$ and $\mathcal{U}$, we define

$$\mathcal{C}[\mathcal{U}] = \{\pi[\alpha_1, \ldots, \alpha_m] \ : \ \pi \in \mathcal{C}_m \text{ and } \alpha_1, \ldots, \alpha_m \in \mathcal{U}\},$$

the inflation of $\mathcal{C}$ by $\mathcal{U}$.

← →

Substitution

Given permutation classes $\mathcal{C}$ and $\mathcal{U}$, we define

$$\mathcal{C}[\mathcal{U}] = \{\pi[\alpha_1, \ldots, \alpha_m] \ : \ \pi \in \mathcal{C}_m \text{ and } \alpha_1, \ldots, \alpha_m \in \mathcal{U}\},$$

the inflation of $\mathcal{C}$ by $\mathcal{U}$.

We also want to inflate classes by themselves.

$$\begin{aligned} \mathcal{C}^{[0]} &= \{1\}, \\ \mathcal{C}^{[1]} &= \mathcal{C}, \\ \mathcal{C}^{[2]} &= \mathcal{C}[\mathcal{C}], \\ &\vdots \\ \mathcal{C}^{[i+1]} &= \mathcal{C}[\mathcal{C}^{[i]}], \\ &\vdots \end{aligned}$$

Substitution

Given permutation classes $\mathcal{C}$ and $\mathcal{U}$, we define

$$\mathcal{C}[\mathcal{U}] = \{\pi[\alpha_1, \ldots, \alpha_m] \ : \ \pi \in \mathcal{C}_m \text{ and } \alpha_1, \ldots, \alpha_m \in \mathcal{U}\},$$

the inflation of $\mathcal{C}$ by $\mathcal{U}$.

We also want to inflate classes by themselves.

$$\begin{aligned} \mathcal{C}^{[0]} &= \{1\}, \\ \mathcal{C}^{[1]} &= \mathcal{C}, \\ \mathcal{C}^{[2]} &= \mathcal{C}[\mathcal{C}], \\ &\vdots \\ \mathcal{C}^{[i+1]} &= \mathcal{C}[\mathcal{C}^{[i]}], \\ &\vdots \end{aligned}$$

The substitution completion of the class $\mathcal{C}$ is

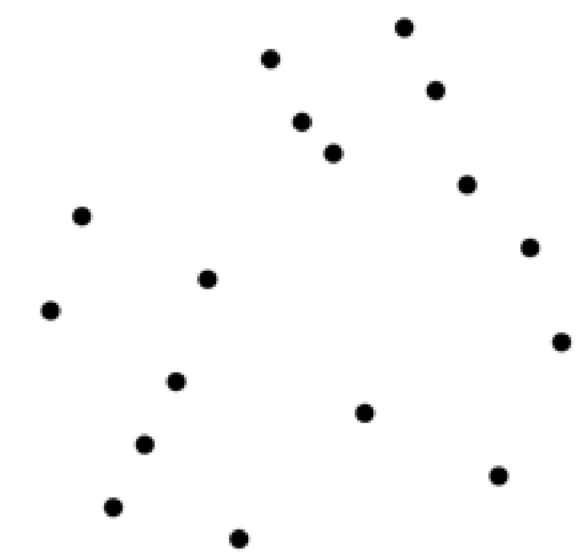
$$\langle \mathcal{C} \rangle = \bigcup_{i \in \mathbb{N}} \mathcal{C}^{[i]}.$$

(Inflate anything in $\mathcal{C}$ by any sequence of permutations in $\mathcal{C}$, as many times as you like.)

Griddings

Let M be a matrix of permutation classes. The permutation π has an M-gridding if π can be chopped up into a block structure ("gridded") such that each block lies in the class specified by M.

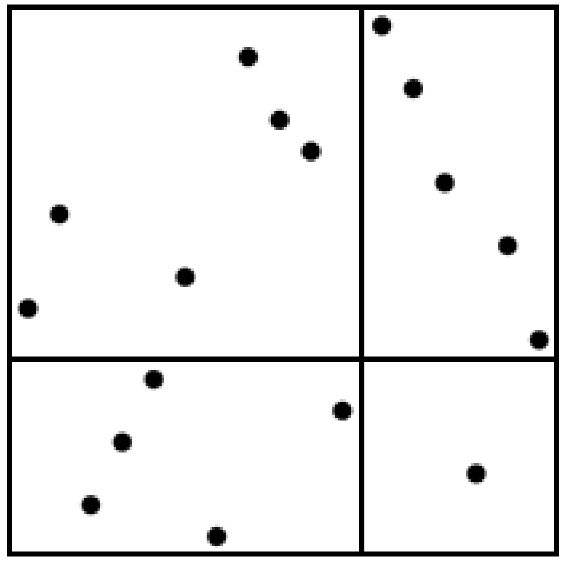
$$M = \begin{pmatrix} \mathrm{Av}(231) & \mathrm{Av}(12) \\ \mathrm{Av}(321) & \mathrm{Av}(12,21) \end{pmatrix}$$



Griddings

Let M be a matrix of permutation classes. The permutation π has an M-gridding if π can be chopped up into a block structure ("gridded") such that each block lies in the class specified by M.

$$M = \begin{pmatrix} \mathrm{Av}(231) & \mathrm{Av}(12) \\ \mathrm{Av}(321) & \mathrm{Av}(12, 21) \end{pmatrix}$$



The grid class of M is defined as

$$\mathrm{Grid}(M) = \{\pi \,:\, \pi \text{ has an } M\text{-gridding}\}.$$

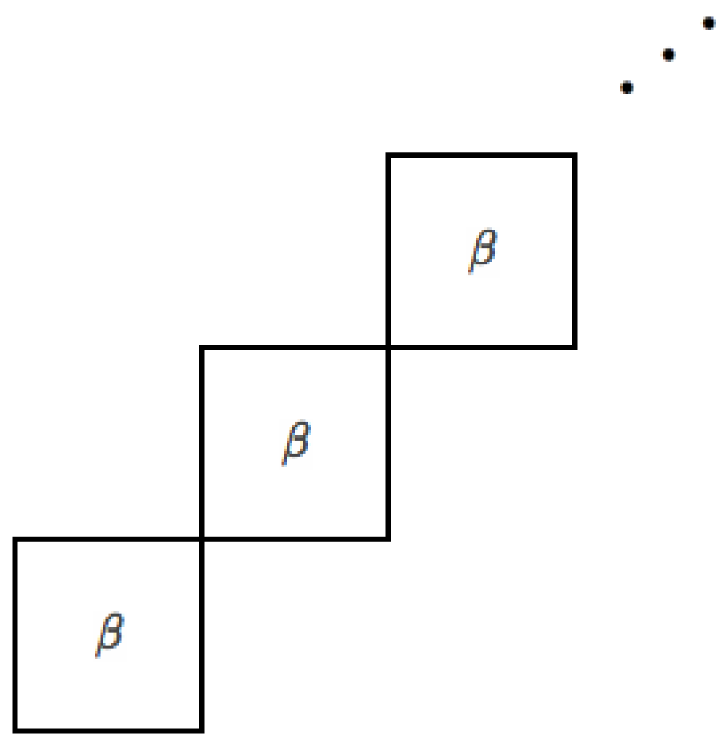
Characterization

Let $\mathcal{C}$ and $\mathcal{D}$ be classes. $\mathcal{C}$ is $\mathcal{D}$-griddable if there is *some* (finite) matrix M, all entries of which equal $\mathcal{D}$, for which $\mathcal{C} \subseteq \mathrm{Grid}(M)$.

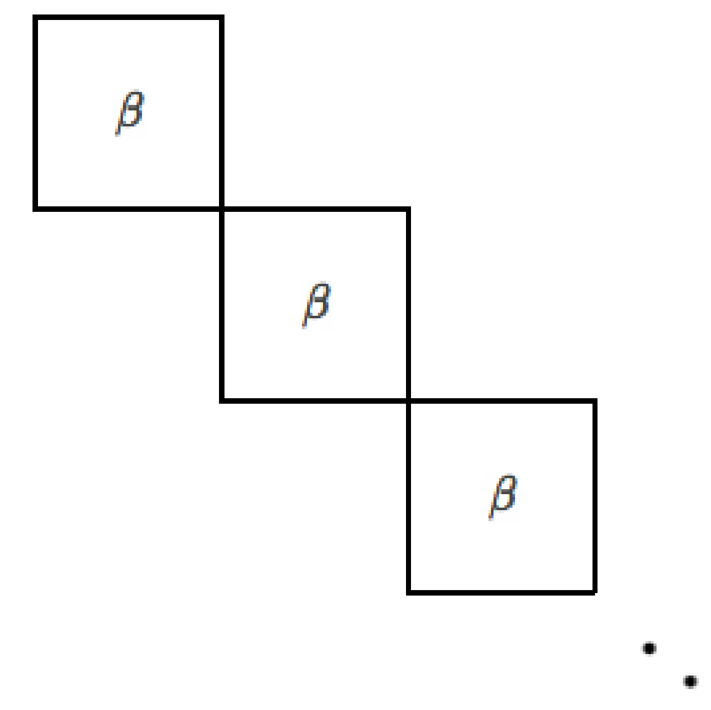
Characterization

Let $\mathcal{C}$ and $\mathcal{D}$ be classes. $\mathcal{C}$ is $\mathcal{D}$-griddable if there is *some* (finite) matrix $\mathcal{M}$, all entries of which equal $\mathcal{D}$, for which $\mathcal{C} \subseteq \mathrm{Grid}(\mathcal{M})$.

Theorem (V 2011). The class $\mathcal{C}$ is $\mathcal{D}$-griddable if and only if $\mathcal{C}$ contains neither arbitrarily long sums nor skew sums of basis elements of $\mathcal{D}$.



a sum



a skew sum

Key take-away: We can tell if a class is $\mathcal{D}$-griddable.

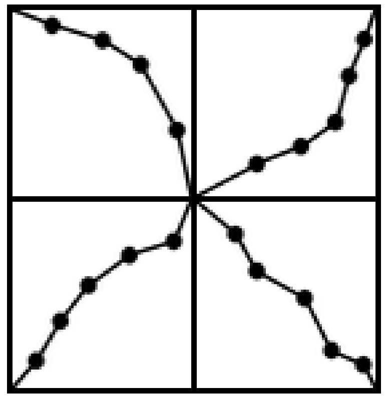
Monotone griddings

$\mathrm{Grid}(M)$ is a monotone grid class if the entries of M are monotone (or empty) classes — $\mathrm{Av}(21)$, $\mathrm{Av}(12)$, or $\emptyset$.

Monotone griddings

$\mathrm{Grid}(M)$ is a monotone grid class if the entries of M are monotone (or empty) classes — $\mathrm{Av}(21)$, $\mathrm{Av}(12)$, or $\emptyset$.

$$M = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(12) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{pmatrix}$$

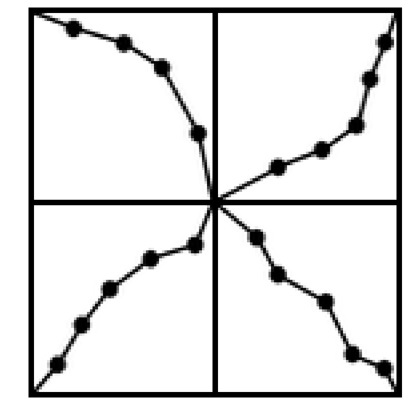


These are the skew-merged permutations, $\mathrm{Av}(2143, 3412)$. They were introduced by Stankova in 1994 and first enumerated in...

Monotone griddings

Grid(M) is a **monotone grid class** if the entries of M are monotone (or empty) classes — Av(21), Av(12), or $\emptyset$.

$$M = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(12) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{pmatrix}$$



These are the **skew-merged permutations**, Av(2143, 3412). They were introduced by Stankova in 1994 and first enumerated in...

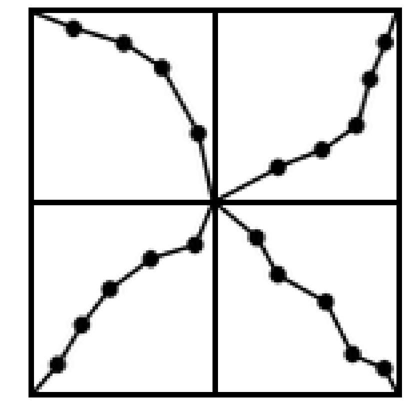
Permutations which are the union of an increasing and a decreasing subsequence

M.D. Atkinson
School of Mathematical and Computational Sciences
North Haugh, St Andrews, Fife KY16 9SS, UK
mda@dcs.st-and.ac.uk

Monotone griddings

$\mathrm{Grid}(M)$ is a monotone grid class if the entries of M are monotone (or empty) classes — $\mathrm{Av}(21)$, $\mathrm{Av}(12)$, or $\emptyset$.

$$M = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(12) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{pmatrix}$$



These are the skew-merged permutations, $\mathrm{Av}(2143, 3412)$. They were introduced by Stankova in 1994 and first enumerated in...

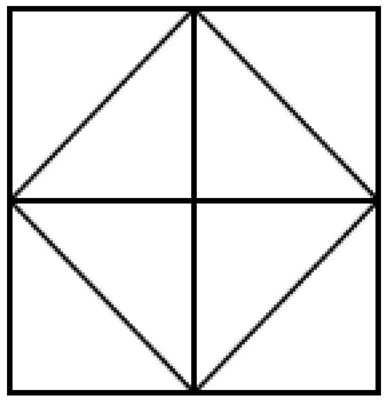
Permutations which are the union of an increasing and a decreasing subsequence

M.D. Atkinson
School of Mathematical and Computational Sciences
North Haugh, St Andrews, Fife KY16 9SS, UK
mda@dcs.st-and.ac.uk

P.S. Henning: The resolution of your conjecture is (basically) in there.

Geometric griddings

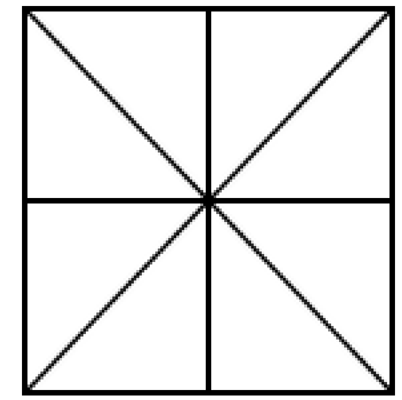
Waton: permutations are "points on a plane".



"Points drawn on a circle"

Waton and V (2011)

$$\frac{1-6x+12x^2-10x^3+5x^4+2x^5-2x^6}{(1-4x+2x^2)(1-x)^3}$$



"The $\mathcal{X}$ class"

Elizalde (2011)

$$\frac{1-3x}{1-4x+2x^2}$$

PERMUTATION PATTERNS 2012

Geometric griddings

Let M be a $t \times u$ $0/\pm 1$ matrix. To construct the standard figure of M, create a $t \times u$ rectangular grid with cells $C_{k,\ell}$ and then:

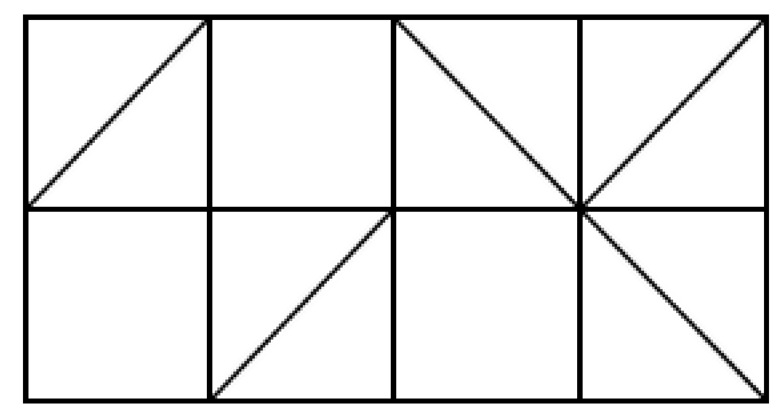
- If $M_{k,\ell} = 1$, draw the SW-NE diagonal in $C_{k,\ell}$.
- If $M_{k,\ell} = -1$, draw the NW-SE diagonal in $C_{k,\ell}$.
- If $M_{k,\ell} = 0$, leave $C_{k,\ell}$ empty.

Geometric griddings

Let M be a $t \times u$ $0/\pm 1$ matrix. To construct the standard figure of M, create a $t \times u$ rectangular grid with cells $C_{k,\ell}$ and then:

- If $M_{k,\ell} = 1$, draw the SW-NE diagonal in $C_{k,\ell}$.
- If $M_{k,\ell} = -1$, draw the NW-SE diagonal in $C_{k,\ell}$.
- If $M_{k,\ell} = 0$, leave $C_{k,\ell}$ empty.

1	0	-1	1
0	1	0	-1

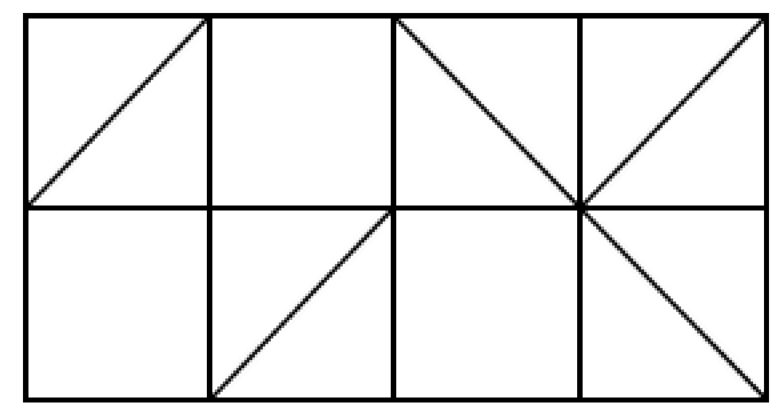


Geometric griddings

Let M be a $t \times u$ $0/\pm 1$ matrix. To construct the standard figure of M, create a $t \times u$ rectangular grid with cells $C_{k,\ell}$ and then:

- If $M_{k,\ell} = 1$, draw the SW-NE diagonal in $C_{k,\ell}$.
- If $M_{k,\ell} = -1$, draw the NW-SE diagonal in $C_{k,\ell}$.
- If $M_{k,\ell} = 0$, leave $C_{k,\ell}$ empty.

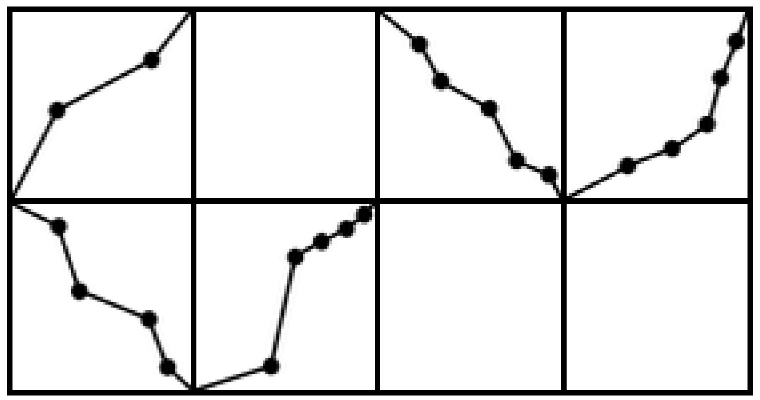
1	0	-1	1
0	1	0	-1

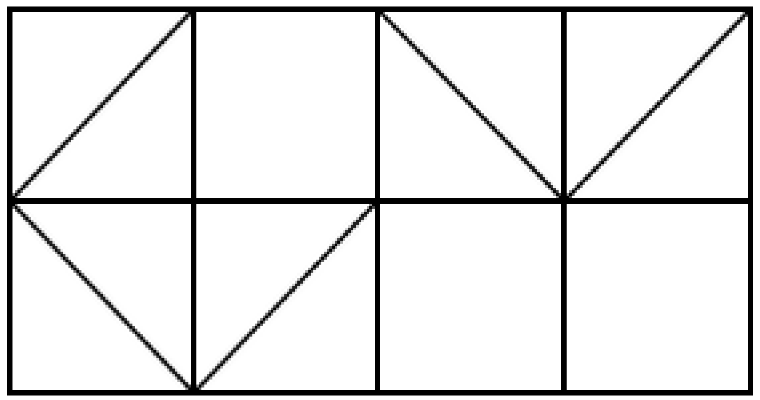


The geometric grid class of M, denoted $\mathrm{Geom}(M)$, is the set of permutations that can be "drawn" on this figure.

"Straightening"

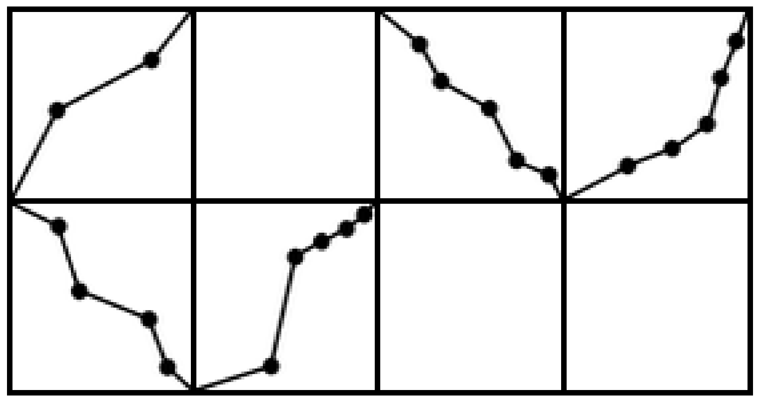
Sometimes we can "straighten" all of the elements of a monotone grid class. In other words, sometimes $\text{Grid}(M) = \text{Geom}(M)$.

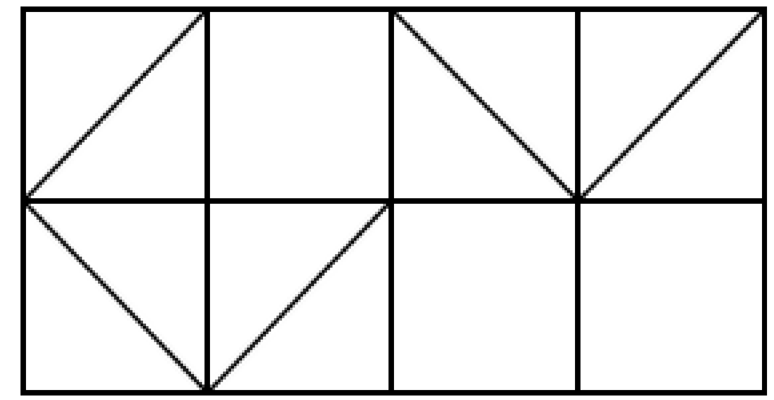




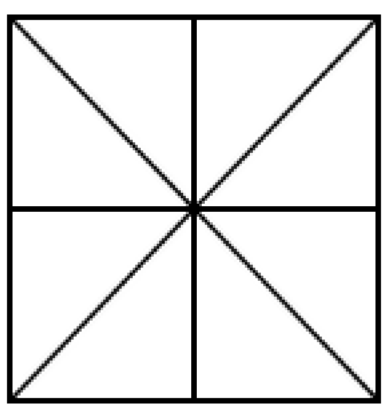
"Straightening"

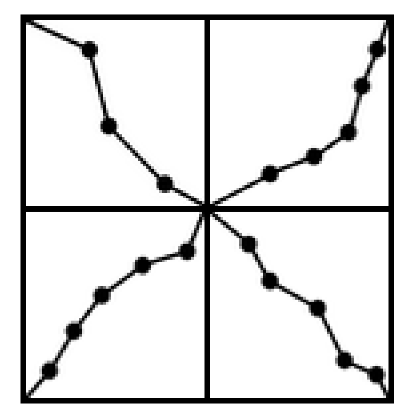
Sometimes we can "straighten" all of the elements of a monotone grid class. In other words, sometimes $\mathrm{Grid}(M) = \mathrm{Geom}(M)$.





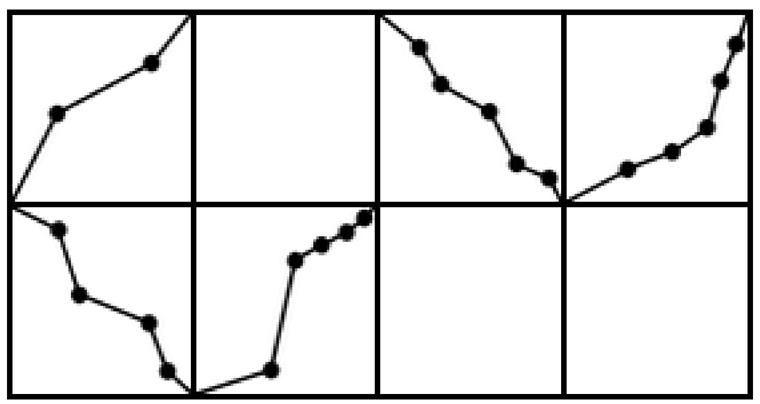
But sometimes we can't; the permutations that can be drawn on an X are a proper subclass of the skew-merged permutations.

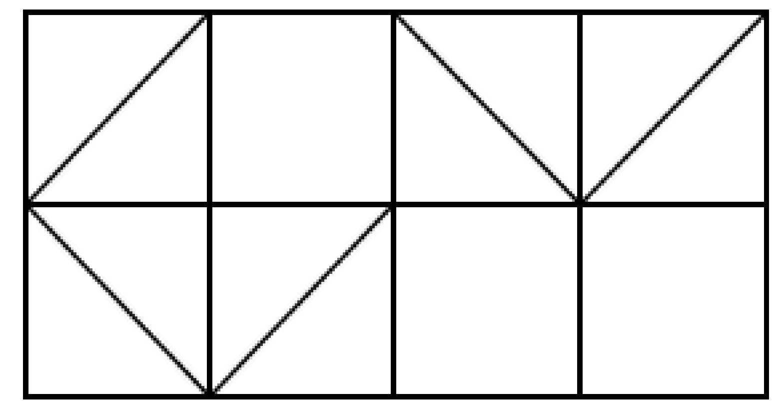




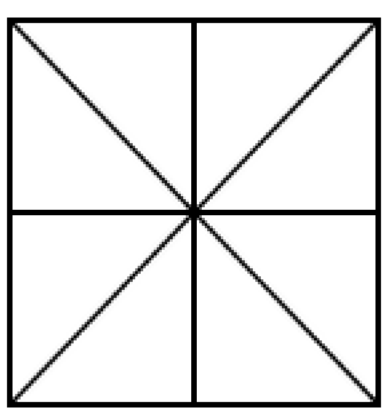
"Straightening"

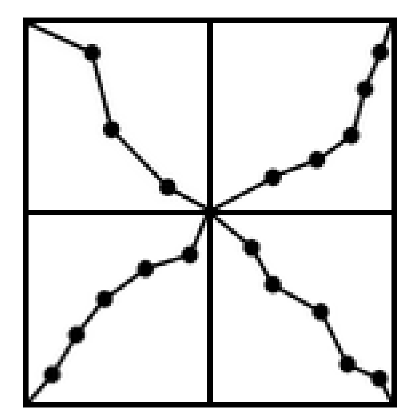
Sometimes we can "straighten" all of the elements of a monotone grid class. In other words, sometimes $\mathrm{Grid}(M) = \mathrm{Geom}(M)$.





But sometimes we can't; the permutations that can be drawn on an X are a proper subclass of the skew-merged permutations.

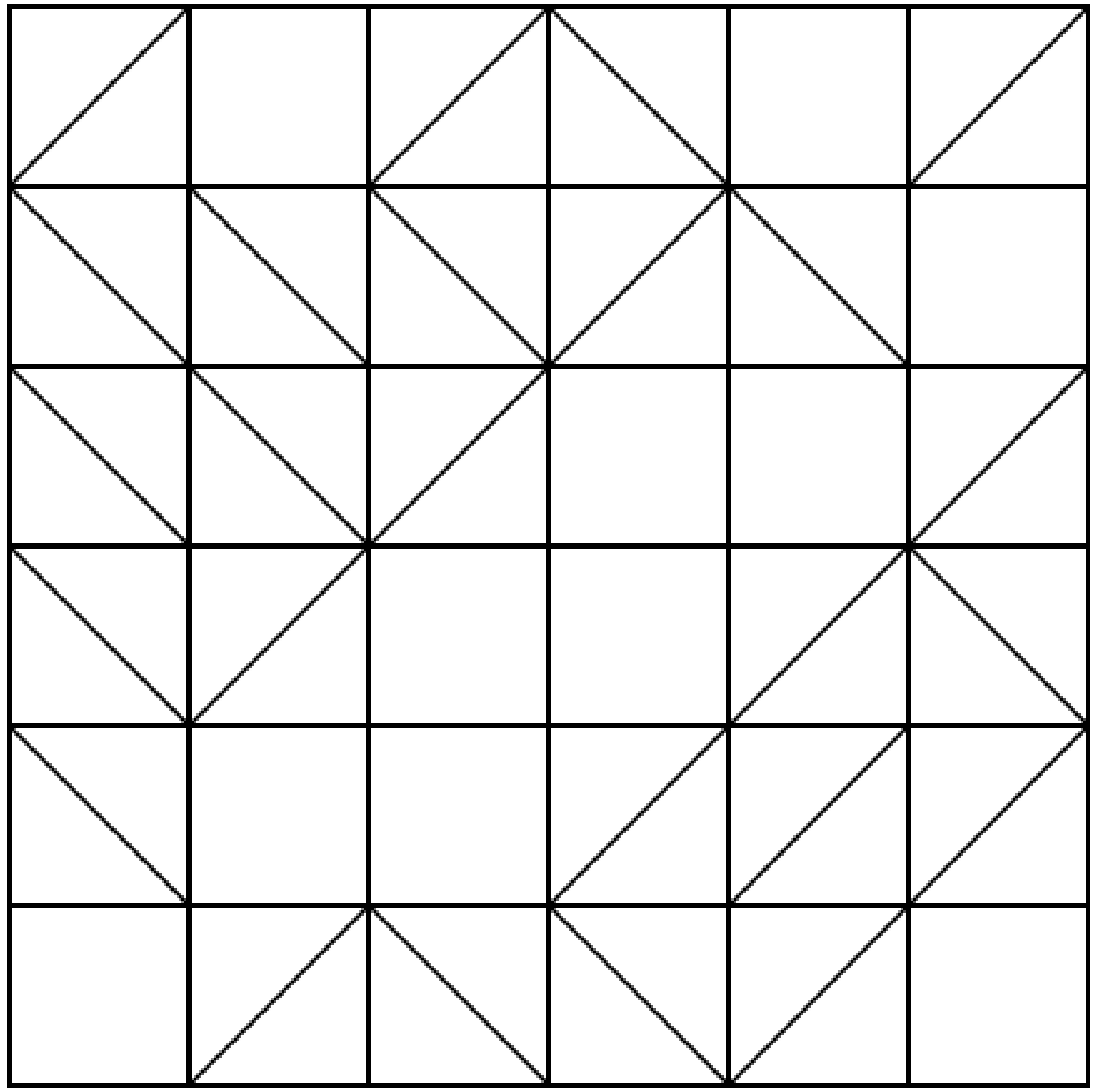




Whether we can straighten a monotone grid class depends on whether it contains cycles.

Geometric griddings

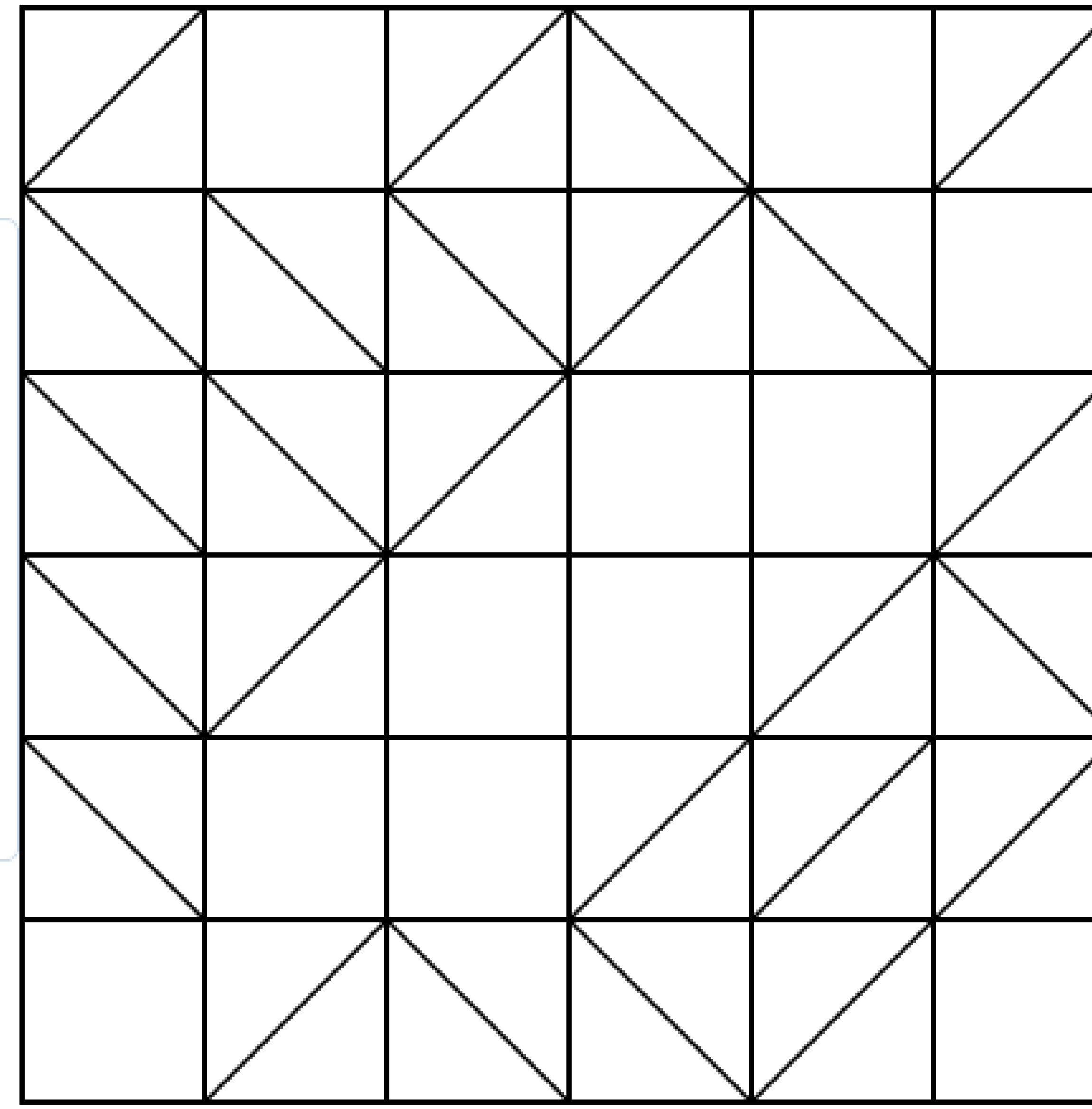
We can encode the elements of any geometric grid class with words from a regular language.



Geometric griddings

We can encode the elements of any geometric grid class with words from a regular language.

> Words are good — Nik Ruškuc PP2010



Enumeration

Let $\mathcal{C}$ be a permutation class.

- $\mathcal{C}$ is strongly rational if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has a rational generating function.
- $\mathcal{C}$ is strongly algebraic if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has an algebraic generating function.

Enumeration

Let $\mathcal{C}$ be a permutation class.

- $\mathcal{C}$ is strongly rational if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has a rational generating function.
- $\mathcal{C}$ is strongly algebraic if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has an algebraic generating function.

1. Theorem (AABRV 2012+). $\mathrm{Geom}(M)$ is strongly rational.

Enumeration

Let $\mathcal{C}$ be a permutation class.

- $\mathcal{C}$ is strongly rational if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has a rational generating function.
- $\mathcal{C}$ is strongly algebraic if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has an algebraic generating function.

1. Theorem (AABRV 2012+). $\mathrm{Geom}(M)$ is strongly rational.
2. Theorem (ARV 2012+). $\langle\mathrm{Geom}(M)\rangle$ is strongly algebraic.

Enumeration

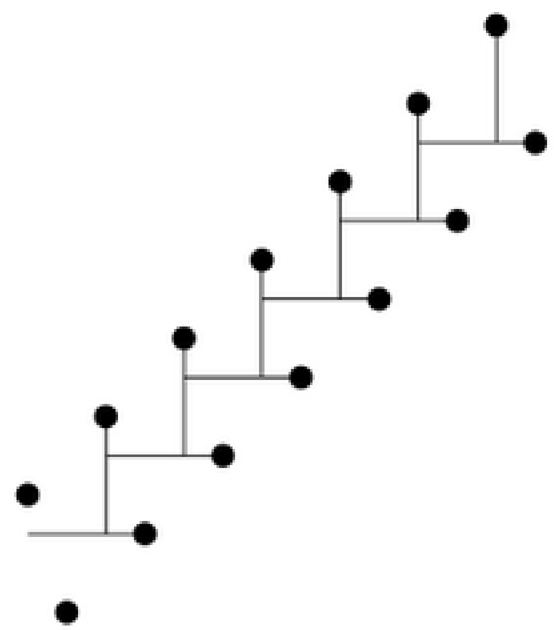
Let $\mathcal{C}$ be a permutation class.

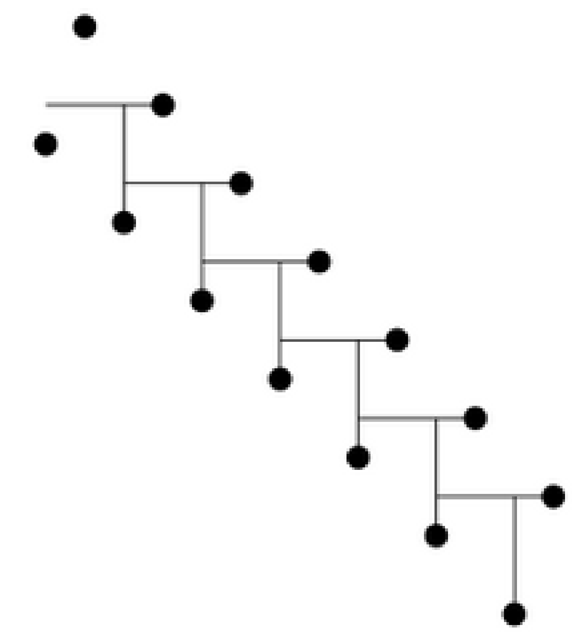
- $\mathcal{C}$ is strongly rational if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has a rational generating function.
- $\mathcal{C}$ is strongly algebraic if every subclass $\mathcal{D} \subseteq \mathcal{C}$ has an algebraic generating function.

1. Theorem (AABRV 2012+). $\mathrm{Geom}(M)$ is strongly rational.
2. Theorem (ARV 2012+). $\langle \mathrm{Geom}(M) \rangle$ is strongly algebraic.
3. Theorem (ARV 2012+). $\mathrm{Geom}(M)[\mathcal{U}]$ is strongly rational whenever $\mathcal{U}$ is strongly rational.

Oscillations

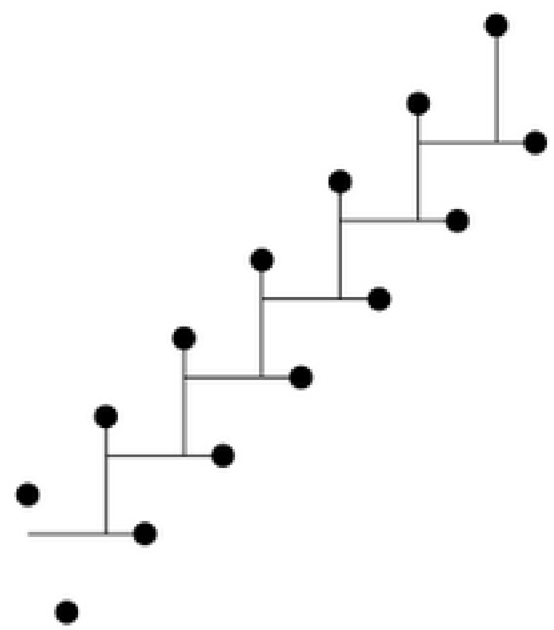
The story of small permutation classes revolves around the oscillations.

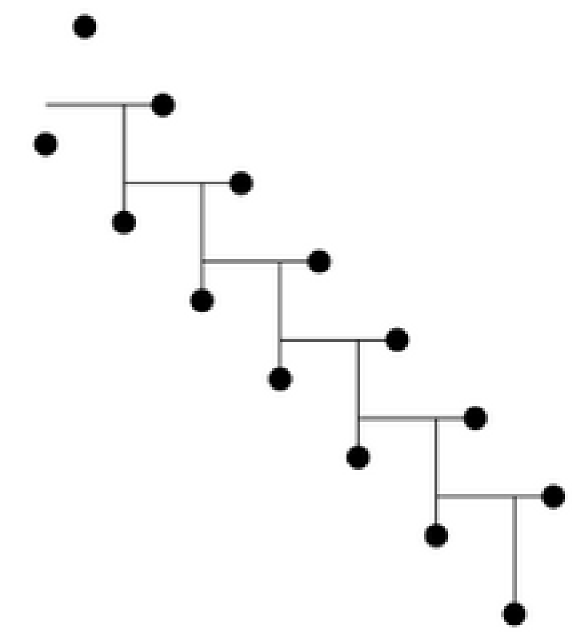




Oscillations

The story of small permutation classes revolves around the oscillations.



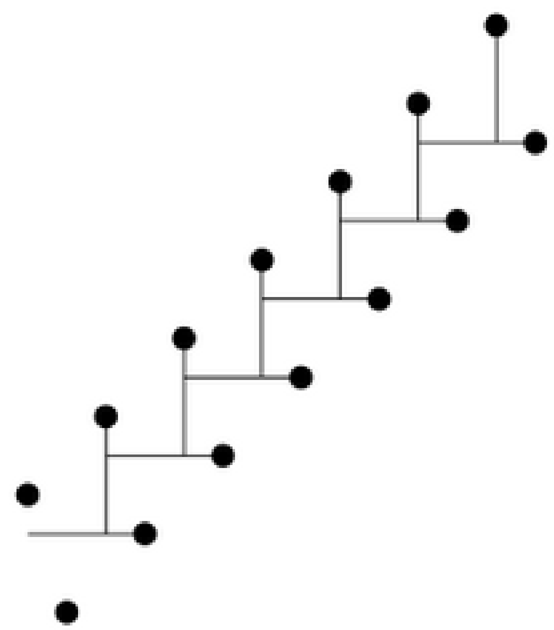


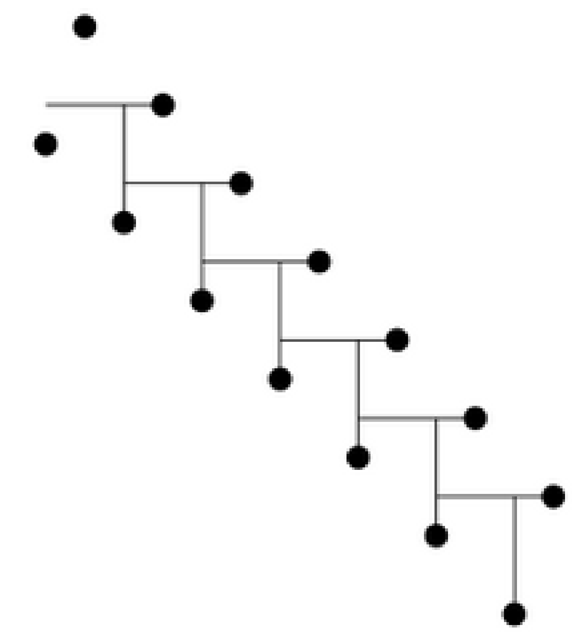
Let $\mathcal{O}$ denote the class of all oscillations, and all permutations contained in oscillations.

It can be computed that $\mathrm{gr}(\mathcal{O}) = \kappa \approx 2.21$.

Oscillations

The story of small permutation classes revolves around the oscillations.





Let $\mathcal{O}$ denote the class of all oscillations, and all permutations contained in oscillations.

It can be computed that $\mathrm{gr}(\mathcal{O}) = \kappa \approx 2.21$.

The basis of $\langle \mathcal{O} \rangle$ can be shown to consist of 25314, 41352, 246153, 251364, 314625, 351624, 415263, and every symmetry of one of these permutations.

Oscillations

The story of small permutation classes revolves around the oscillations.

Let $\mathcal{O}$ denote the class of all oscillations, and all permutations contained in oscillations.

It can be computed that $\mathrm{gr}(\mathcal{O}) = \kappa \approx 2.21$.

The basis of $\langle \mathcal{O} \rangle$ can be shown to consist of 25314, 41352, 246153, 251364, 314625, 351624, 415263, and every symmetry of one of these permutations.

From this, it can be computed that every class with $\mathrm{gr} < 2.24$ is $\langle \mathcal{O} \rangle$-griddable.

Gridding small classes

$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$

Gridding small classes

- $\mathrm{gr}(\mathcal{O}) = \kappa$, so a small class cannot contain all of $\mathcal{O}$.

$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$

Gridding small classes

- $\mathrm{gr}(\mathcal{O}) = \kappa$, so a small class cannot contain all of $\mathcal{O}$.
- Because of the structure of $\mathcal{O}$, this implies that there is a bound, say k, on the length of an oscillation in any small class. Define

$$\mathcal{O}_k = \{\text{oscillations of length at most } k\}$$

$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$

Gridding small classes

- $\mathrm{gr}(\mathcal{O}) = \kappa$, so a small class cannot contain all of $\mathcal{O}$.
- Because of the structure of $\mathcal{O}$, this implies that there is a bound, say k, on the length of an oscillation in any small class. Define
$$\mathcal{O}_k = \{\text{oscillations of length at most } k\}$$
- Every small class is $\langle\mathcal{O}_k\rangle$-griddable for some k.

$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$
$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$	$\langle\mathcal{O}\rangle$

Gridding small classes

$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$

Gridding small classes

- "Deep" inflations make for large permutation classes, so there must be an absolute bound on the "substitution depth" of permutations in a small class.

$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$

Gridding small classes

- "Deep" inflations make for large permutation classes, so there must be an absolute bound on the "substitution depth" of permutations in a small class.
- Define
$$\tilde{\mathcal{O}}_k = \mathcal{O}_k \cup \mathrm{Av}(12) \cup \mathrm{Av}(21).$$

$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$

Gridding small classes

- "Deep" inflations make for large permutation classes, so there must be an absolute bound on the "substitution depth" of permutations in a small class.
- Define
$$\tilde{\mathcal{O}}_k = \mathcal{O}_k \cup \mathrm{Av}(12) \cup \mathrm{Av}(21).$$
- Every small class is $\tilde{\mathcal{O}}_k^{[d]}$-griddable for some d.

$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$
$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$	$\langle \mathcal{O}_k \rangle$

Gridding small classes

$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$

(Grid may have been enlarged.)

Gridding small classes

- Now that the cells have finitely many simple permutations and bounded substitution depth, it is possible to "slice" the gridding. In particular, we can "slice" the griddings of small classes.

$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$

(Grid may have been enlarged.)

Gridding small classes

- Now that the cells have finitely many simple permutations and bounded substitution depth, it is possible to "slice" the gridding. In particular, we can "slice" the griddings of small classes.
- If three cells in the same row or column were to have "unbounded alternations", it would force the growth rate above 3.

$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$

(Grid may have been enlarged.)

Gridding small classes

- Now that the cells have finitely many simple permutations and bounded substitution depth, it is possible to "slice" the gridding. In particular, we can "slice" the griddings of small classes.
- If three cells in the same row or column were to have "unbounded alternations", it would force the growth rate above 3.
- If three cells in a "hook shape" were to have "unbounded alternations", it would force the growth rate above $1 + \phi \approx 2.62$.

$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$	$\tilde{\mathcal{O}}_k^{[d]}$

(Grid may have been enlarged.)

Gridding small classes

- Now that the cells have finitely many simple permutations and bounded substitution depth, it is possible to "slice" the gridding. In particular, we can "slice" the griddings of small classes.
- If three cells in the same row or column were to have "unbounded alternations", it would force the growth rate above 3.
- If three cells in a "hook shape" were to have "unbounded alternations", it would force the growth rate above $1 + \phi \approx 2.62$.
- So we have only bounded alternations, and can "slice" them.

$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$
$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$	$\tilde{O}_k^{[d]}$

(Grid may have been enlarged.)

Gridding small classes

	$\tilde{\mathcal{O}}_k^{[d]}$				
		$\tilde{\mathcal{O}}_k^{[d]}$			
	$\tilde{\mathcal{O}}_k^{[d]}$				
					$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$			$\tilde{\mathcal{O}}_k^{[d]}$		
				$\tilde{\mathcal{O}}_k^{[d]}$	

(Grid may have been enlarged.)

Gridding small classes

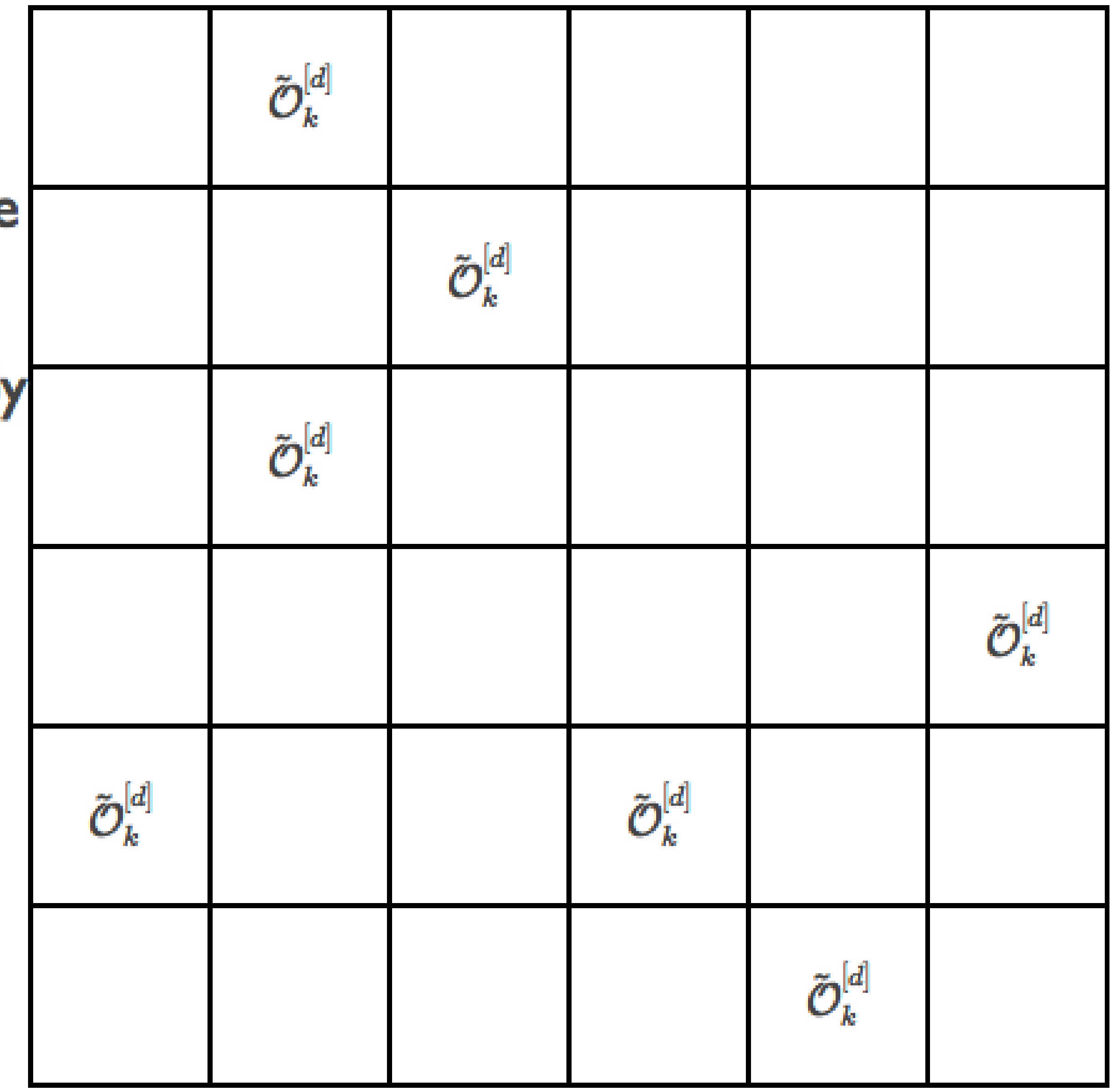
- If a non-monotone cell were to have "unbounded alternations" with another (even monotone cell) in the same row or column, that would force the growth rate above $1+\sqrt{2} \approx 2.41$.

	$\tilde{\mathcal{O}}_k^{[d]}$				
		$\tilde{\mathcal{O}}_k^{[d]}$			
	$\tilde{\mathcal{O}}_k^{[d]}$				
					$\tilde{\mathcal{O}}_k^{[d]}$
$\tilde{\mathcal{O}}_k^{[d]}$			$\tilde{\mathcal{O}}_k^{[d]}$		
				$\tilde{\mathcal{O}}_k^{[d]}$	

(Grid may have been enlarged.)

Gridding small classes

- If a non-monotone cell were to have "unbounded alternations" with another (even monotone cell) in the same row or column, that would force the growth rate above $1+\sqrt{2}\approx 2.41$.
- So we can slice these bounded alternations, and thereby insist that only monotone cells can share a row or column.



(Grid may have been enlarged.)

Gridding small classes

Theorem (V 2011). Every small permutation class is M-griddable for a matrix M in which:

1. every entry is $\tilde{\mathcal{O}}_k^{[d]}$, $\mathrm{Av}(21)$, $\mathrm{Av}(12)$, or the empty set;
2. every entry equal to $\tilde{\mathcal{O}}_k^{[d]}$ is the unique nonempty entry in its row and column; and
3. if two nonempty entries share a row or a column with each other then neither shares a row or column with any other nonempty entry.

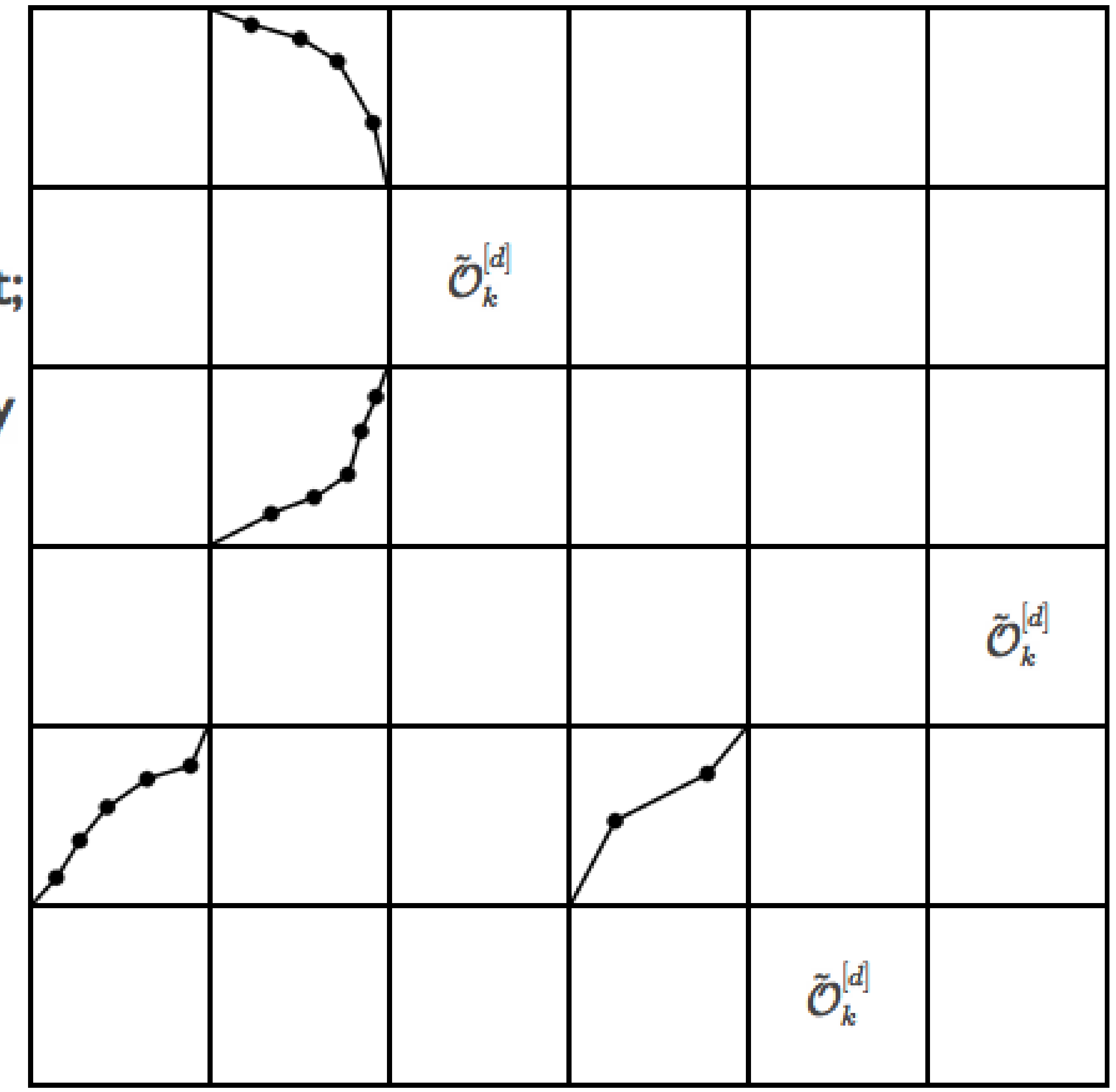
	$\mathrm{Av}(12)$				
		$\tilde{\mathcal{O}}_k^{[d]}$			
	$\mathrm{Av}(21)$				
					$\tilde{\mathcal{O}}_k^{[d]}$
$\mathrm{Av}(21)$			$\mathrm{Av}(21)$		
				$\tilde{\mathcal{O}}_k^{[d]}$	

(Grid may have been enlarged.)

Counting small classes

Theorem (V 2011). Every small permutation class is M-griddable for a matrix M in which:

1. every entry is $\tilde{\mathcal{O}}_k^{[d]}$, $\mathrm{Av}(21)$, $\mathrm{Av}(12)$, or the empty set;
2. every entry equal to $\tilde{\mathcal{O}}_k^{[d]}$ is the unique nonempty entry in its row and column; and
3. if two nonempty entries share a row or a column with each other then neither shares a row or column with any other nonempty entry.



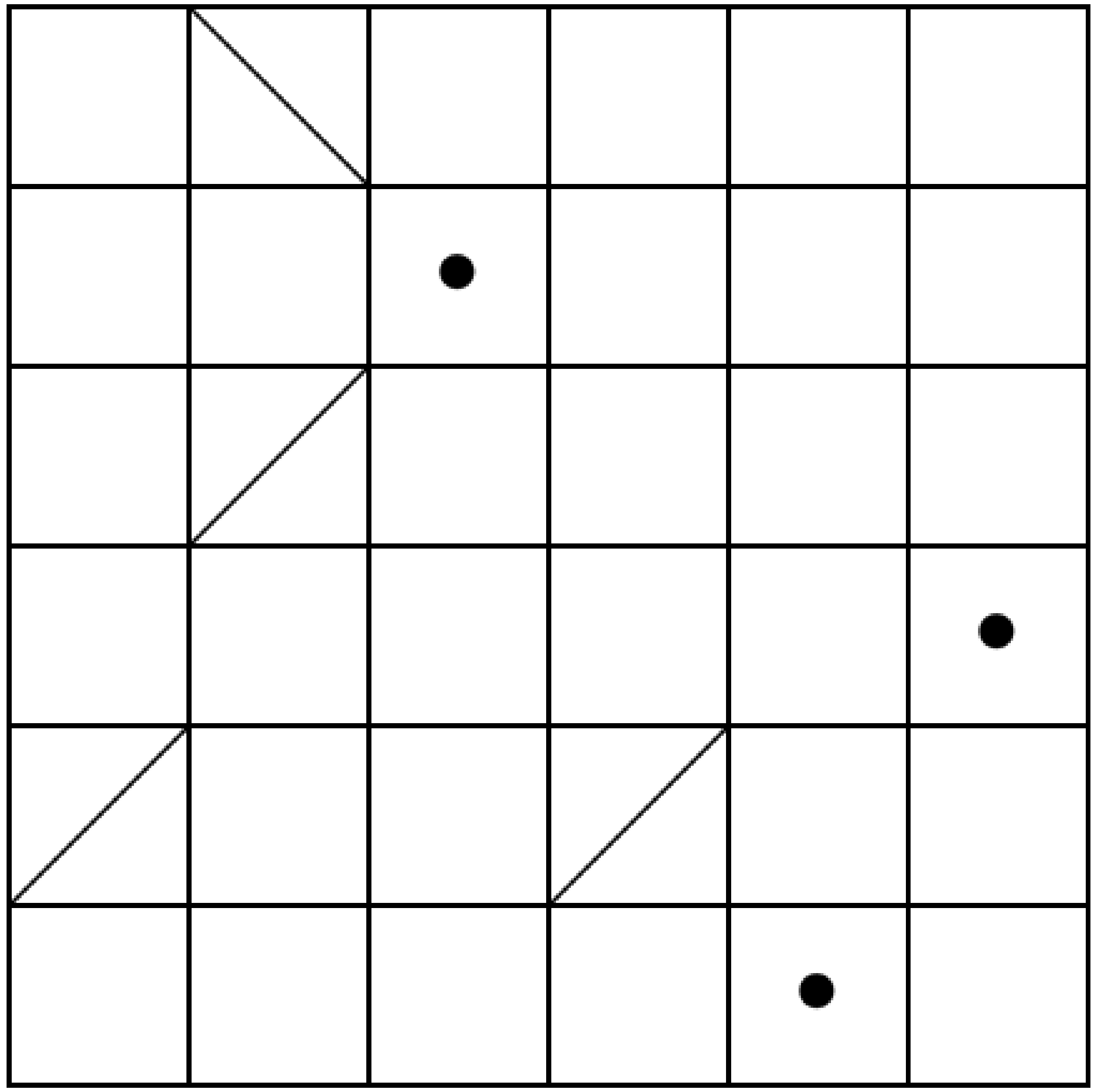
Counting small classes

Now we're looking at the inflation of a geometric grid class by $\tilde{\mathcal{O}}_k^{[d]}$.

$\tilde{\mathcal{O}}_k$ contains only finitely many non-monotone permutations, so it is geometrically griddable itself.

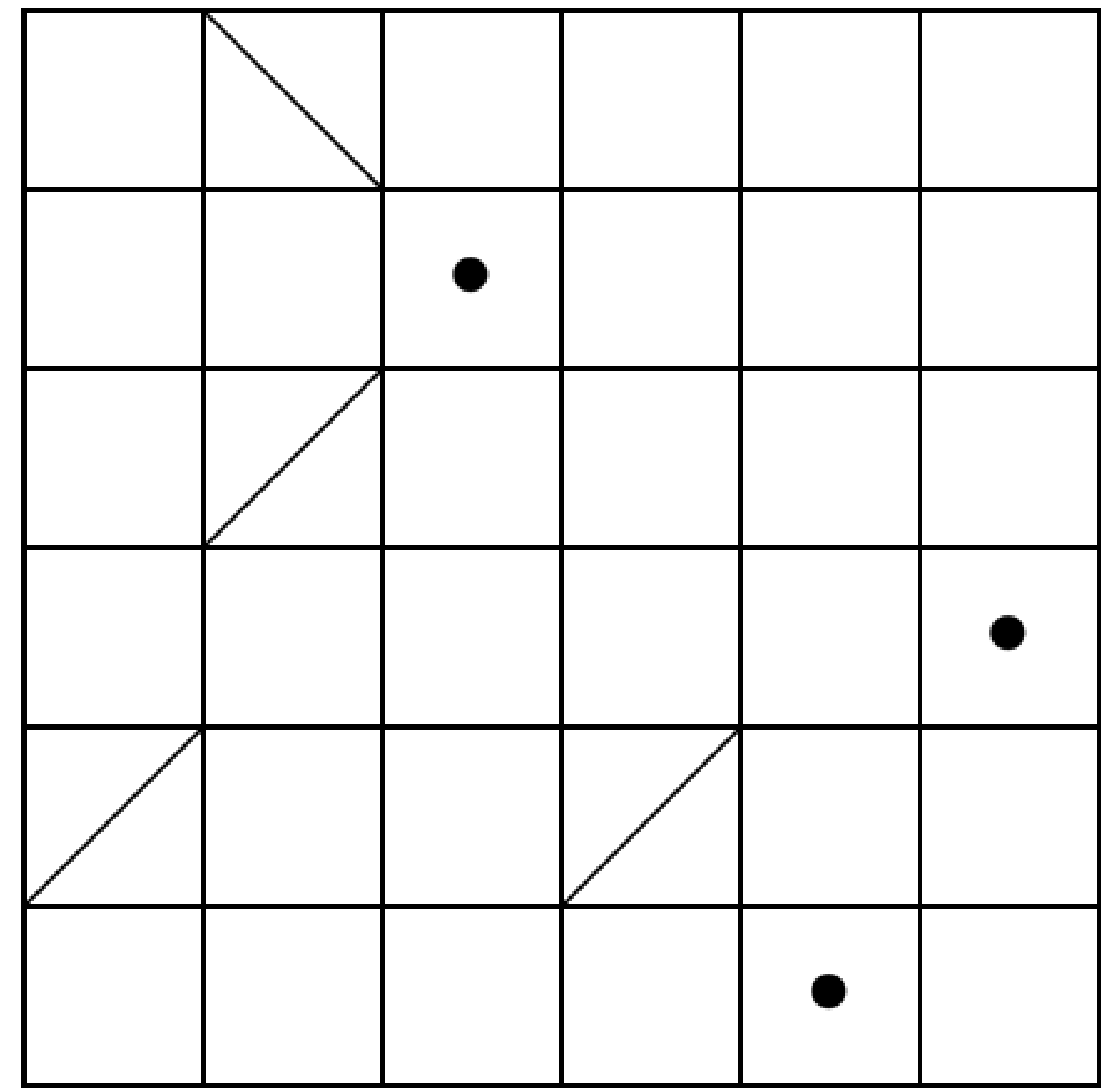
Therefore:

- $\tilde{\mathcal{O}}_k$ is strongly rational,
- $\tilde{\mathcal{O}}_k^{[2]} = \tilde{\mathcal{O}}_k[\tilde{\mathcal{O}}_k]$ is strongly rational, ...
- $\tilde{\mathcal{O}}_k^{[d]} = \tilde{\mathcal{O}}_k[\tilde{\mathcal{O}}_k^{[d-1]}]$ is strongly rational, so
- $\mathrm{Geom}(M)[\tilde{\mathcal{O}}_k^{[d]}]$ is strongly rational.



Counting small classes

Theorem (ARV 2012+). If $\mathrm{gr}(\mathcal{C}) < \kappa \approx 2.21$, then $\mathcal{C}$ has a rational generating function.



Other applications

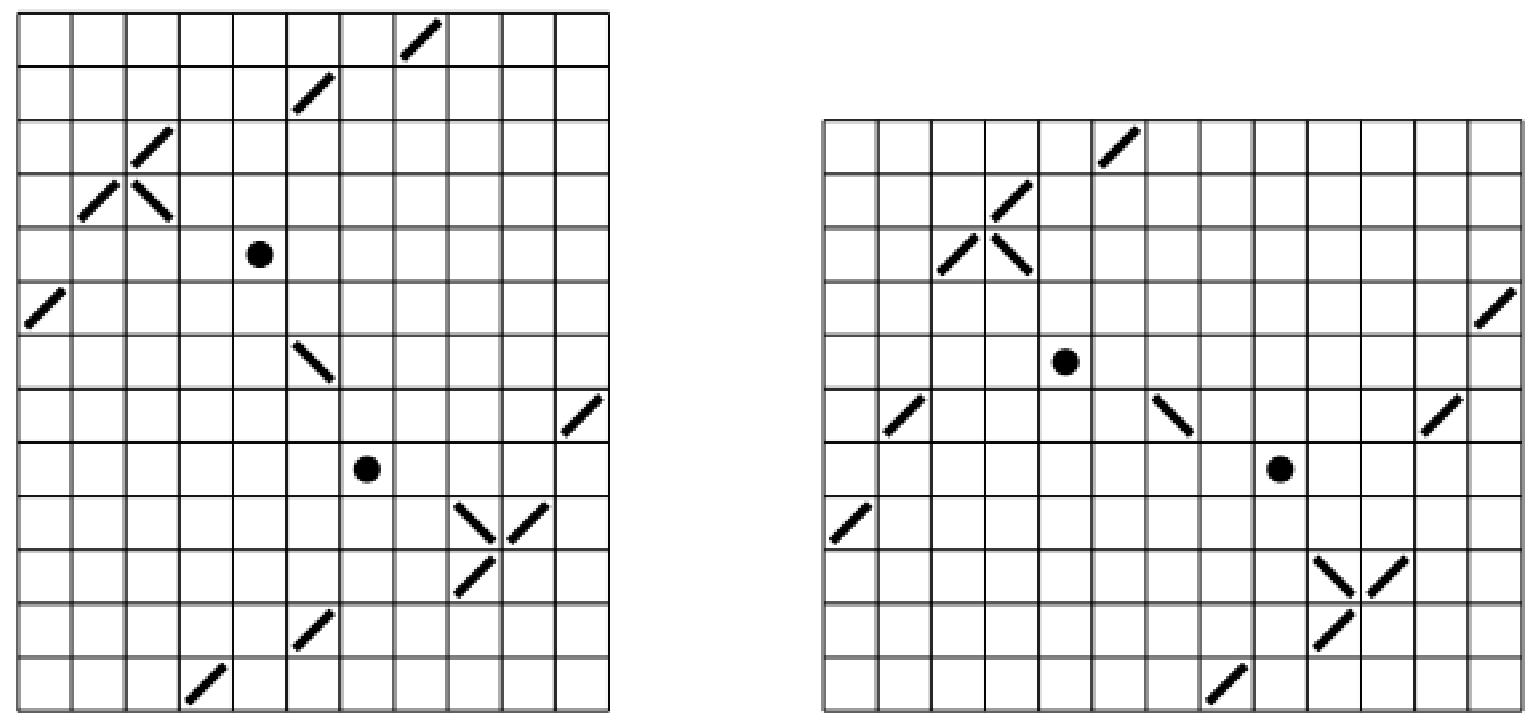
Classes avoiding two patterns of length 4

There are 56 symmetry classes and 38 Wilf equivalence classes, of which 18 have been enumerated.

B	sequence enumerating $Av_n(B)$	OEIS	type of sequence	exact enumeration reference
4321, 1234	1, 2, 6, 22, 86, 306, 882, 1764, ...	n/a	finite	Erdős–Szekeres theorem
4312, 1234	1, 2, 6, 22, 86, 321, 1085, 3266, ...	A116705	polynomial	Kremer & Shiu (2003)
4321, 3124	1, 2, 6, 22, 86, 330, 1198, 4087, ...	A116708	rational g.f.	Kremer & Shiu (2003)
4312, 2134	1, 2, 6, 22, 86, 330, 1206, 4174, ...	A116706	rational g.f.	Kremer & Shiu (2003)
4321, 1324	1, 2, 6, 22, 86, 332, 1217, 4140, ...	A165524	polynomial	Vatter (2012)
4321, 2143	1, 2, 6, 22, 86, 333, 1235, 4339, ...	A165525		
4312, 1324	1, 2, 6, 22, 86, 335, 1266, 4598, ...	A165526		
4231, 2143	1, 2, 6, 22, 86, 335, 1271, 4680, ...	A165527	rational g.f.	Albert, Atkinson & Brignall (2011)
4231, 1324	1, 2, 6, 22, 86, 336, 1282, 4758, ...	A165528	rational g.f.	Albert, Atkinson & Vatter (2009)
4213, 2341	1, 2, 6, 22, 86, 336, 1290, 4870, ...	A116709	rational g.f.	Kremer & Shiu (2003)
4312, 2143	1, 2, 6, 22, 86, 337, 1295, 4854, ...	A165529		
4213, 1243	1, 2, 6, 22, 86, 337, 1299, 4910, ...	A116710	rational g.f.	Kremer & Shiu (2003)

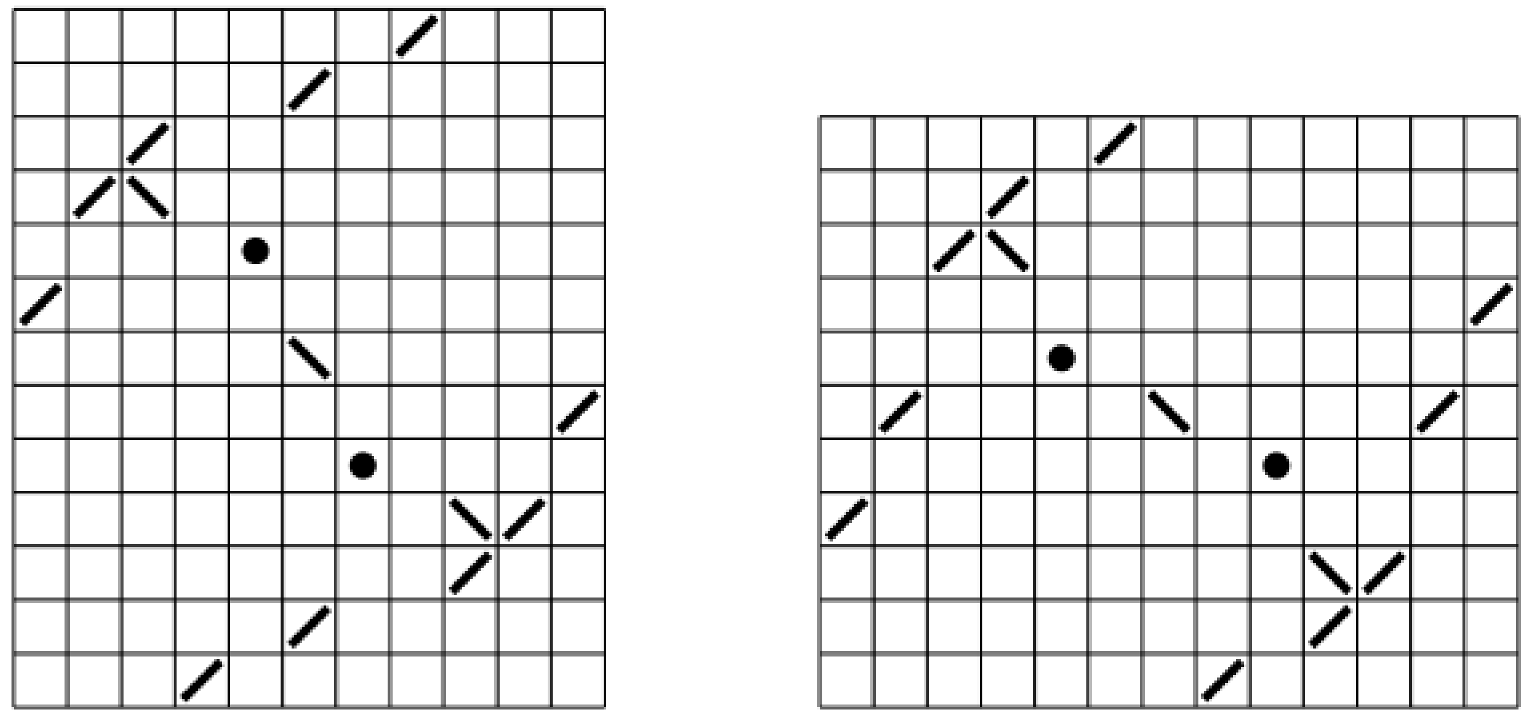
Other applications

Theorem (Albert, Atkinson, and Brignall 2011). The class $\mathrm{Av}(2143, 4231)$ is the union of two geometric grid classes:



Other applications

Theorem (Albert, Atkinson, and Brignall 2011). The class $\mathrm{Av}(2143, 4231)$ is the union of two geometric grid classes:

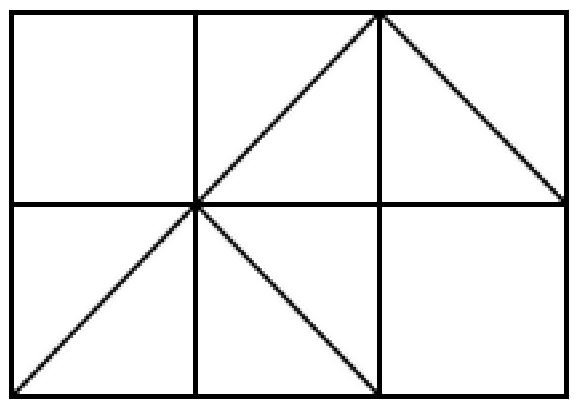


The generating function of this class is

$$\frac{x - 11x^2 + 51x^3 - 126x^4 + 186x^5 - 165x^6 + 87x^7 - 23x^8 + 3x^9}{(1-3x)(1-x)^4(1-3x+x^2)^2}.$$

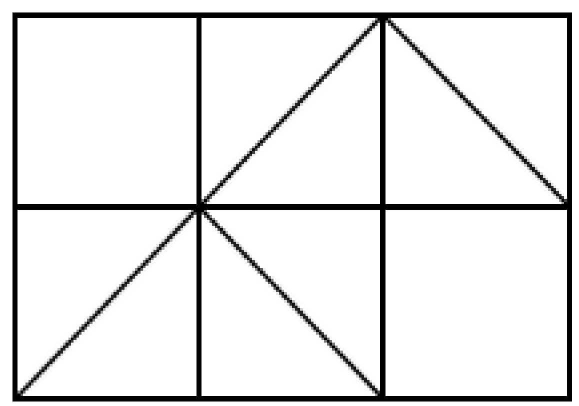
Other applications

Theorem (Albert, Atkinson, and V 2012+). The class $\mathrm{Av}(4231, 3124)$ is contained in the substitution completion of the geometric grid class



Other applications

Theorem (Albert, Atkinson, and V 2012+). The class $\mathrm{Av}(4231, 3124)$ is contained in the substitution completion of the geometric grid class

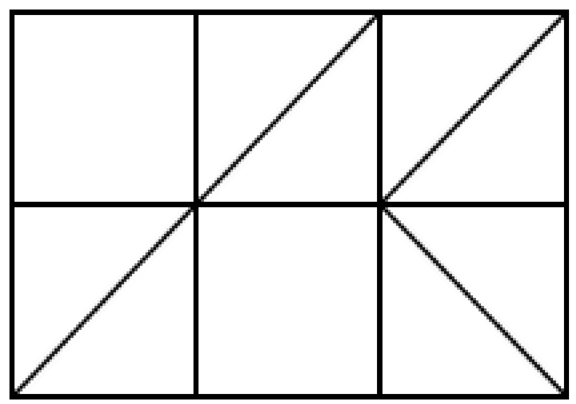


Its generating function is

$$\frac{1-8x+20x^2-20x^3+10x^4-2x^5-\sqrt{1-12x+52x^2-96x^3+68x^4-16x^5}}{2(1-3x+x^2)(-1+5x-4x^2+x^3)}.$$

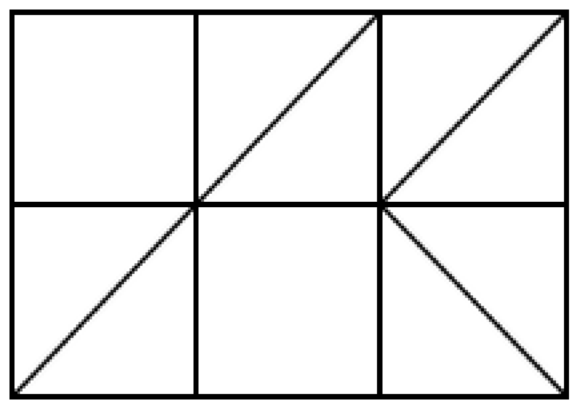
Other applications

Theorem (Albert, Atkinson, and V 2012+). The class $\mathrm{Av}(4312, 3142)$ is contained in the substitution completion of the geometric grid class



Other applications

Theorem (Albert, Atkinson, and V 2012+). The class $\mathrm{Av}(4312, 3142)$ is contained in the substitution completion of the geometric grid class

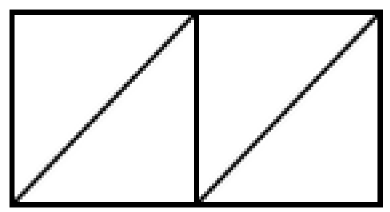


Its generating function satisfies the algebraic equation

$$\begin{aligned}(x^3-2x^2+x)f^4 \quad &+ \quad (4x^3-9x^2+6x-1)f^3\\ &+ \quad (6x^3-12x^2+7x-1)f^2\\ &+ \quad (4x^3-5x^2+x)f\\ &+ \quad x^3 \qquad\qquad\qquad\qquad = \quad 0.\end{aligned}$$

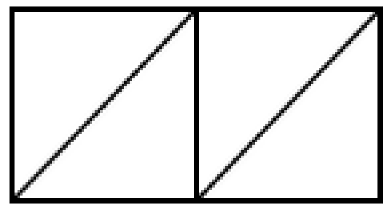
Other applications

Theorem (Albert, Atkinson, and V 2012+). The class $\mathrm{Av}(4213, 3142)$ is contained in the substitution completion of the geometric grid class



Other applications

Theorem (Albert, Atkinson, and V 2012+). The class $\mathrm{Av}(4213, 3142)$ is contained in the substitution completion of the geometric grid class



Its generating function satisfies the algebraic equation

$$\begin{aligned} x^3 f^6 &+ (7x^3 - 7x^2 + 2x) f^5 \\ &+ (x^4 + 14x^3 - 21x^2 + 10x - 1) f^4 \\ &+ (4x^4 + 8x^3 - 19x^2 + 11x - 2) f^3 \\ &+ (6x^4 - 5x^3 - 2x^2 + 2x) f^2 \\ &+ (4x^4 - 7x^3 + 4x^2 - x) f \\ &+ x^4 - 2x^3 + x^2 = 0. \end{aligned}$$

The future

The future

For PP2013?

Conjecture (Albert and Atkinson 2005). Every proper, finitely based, subclass of $\mathrm{Av}(321)$ has a rational generating function.

PERMUTATION PATTERNS 2012

The future

For PP2013?

Conjecture (Albert and Atkinson 2005). Every proper, finitely based, subclass of $\mathrm{Av}(321)$ has a rational generating function.

For PP2020?

Fix the gap in the number line...

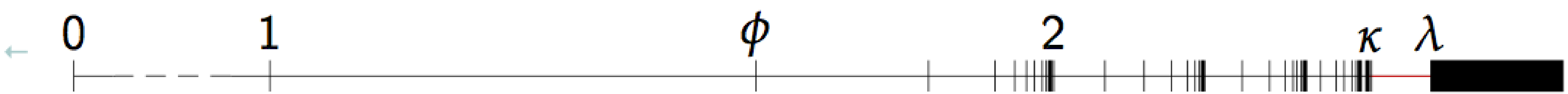
The future

For PP2013?

Conjecture (Albert and Atkinson 2005). Every proper, finitely based, subclass of $\mathrm{Av}(321)$ has a rational generating function.

For PP2020?

Fix the gap in the number line...



Thank you!