Permuted Basement Fillings, k-ary Trees, and Watermelons

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Schur Functions

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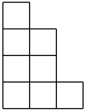
$dg(\lambda) =$  $\lambda = (3, 2, 2, 1)$

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$$T = \begin{array}{|c|c|c|} \cline{1-1} 5 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 1 & 4 \\ \hline \end{array} \qquad \lambda = (3, 2, 2, 1)$$

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$\longrightarrow$

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2			
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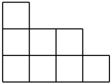
Diagrams

Partition



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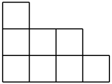

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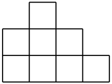

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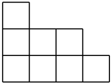

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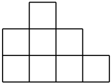

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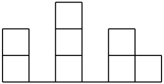
Weak Composition



$\gamma = (2, 0, 3, 0, 2, 1)$

Nonsymmetric Macdonald Polynomials

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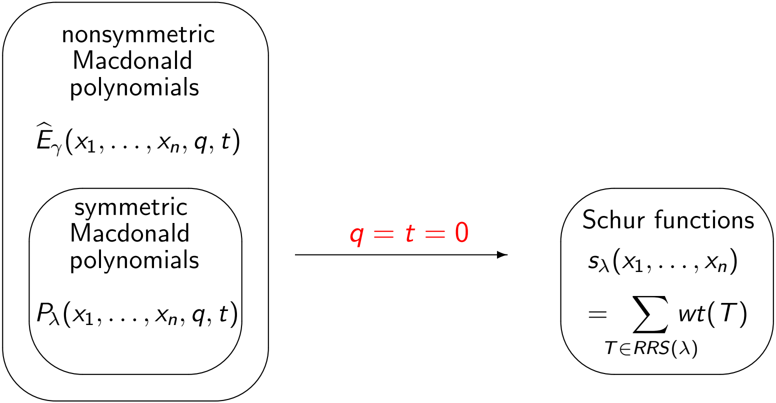

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Demazure atoms

$$\widehat{E}^{id}_\gamma(x_1, \ldots, x_n) = \sum_{F \in PBF(\gamma, id)} wt(F)$$

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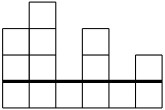
What is a PBF?

Let γ be a weak composition of m into n parts.



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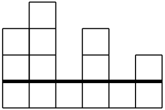

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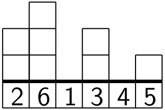

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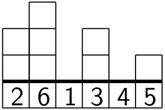

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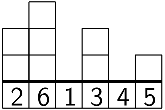

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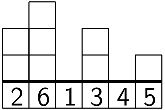

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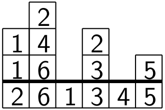

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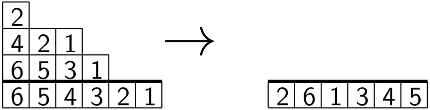

Place each number in the leftmost position possible, so long as columns weakly decrease reading up.

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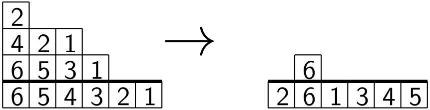

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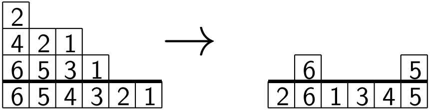

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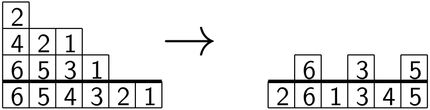

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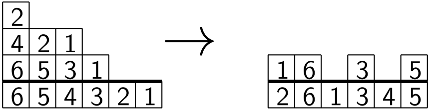

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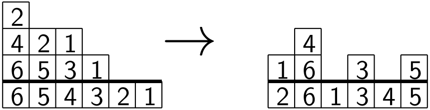

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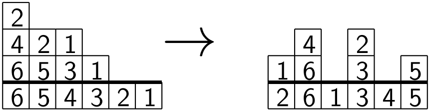

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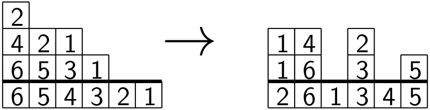

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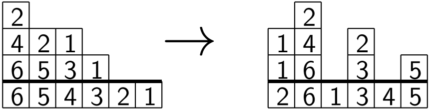

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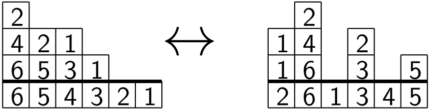

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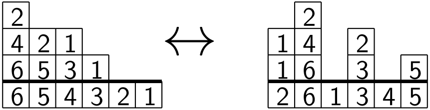

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Let $PBF(\gamma, \sigma) = \{$PBFs F of shape γ and basement $\sigma\}$.

Algebra

Haglund, Mason and Remmel defined an insertion procedure for PBFs which

- takes a PBF and a number to insert, adding a single cell to the top of a column, to create another PBF
- generalizes the Robinson Schensted Knuth insertion procedure for tableaux
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Definition

$$\widehat{E}_{\gamma}^{\sigma}(x_1, \ldots, x_n) = \sum_{F \in PBF(\gamma,\sigma)} wt(F)$$

For any fixed σ, can always write $s_\lambda = \sum_{\gamma} \widehat{E}_{\gamma}^{\sigma}$.

Enumeration

For a fixed λ, $\#RRS(\lambda)$ can be obtained by the hook formula. Is there an analogous way to count PBFs of a given shape?

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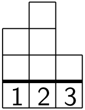
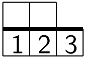


$\gamma_1 = (2, 3, 1)$



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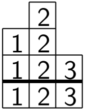

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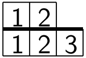
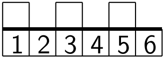


$\gamma_2 = (1, 1, 0)$

$\gamma = (1,0)^k$



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1		3		5	
1	2	3	4	5	6

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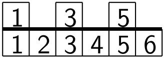

Theorem

$\#PBF((1,0)^k, id) = 1$

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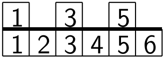

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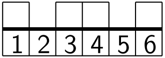

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$\gamma = (1,0)^k$

1		3		5	
1	2	3	4	5	6

Theorem

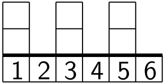
$\#PBF((1,0)^k, id) = 1$

1		3	4		6
1	2	3	4	5	6

Theorem

Let γ be such that $\gamma_i \leq 1$ for all i. Then $\#PBF(\gamma, id) = 1$

$\gamma = (2,0)^k$



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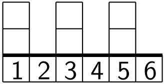

k	1	2	3	4	5
$\#PBF((2,0)^k, id)$	1	2	5	14	42

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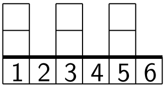

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$\#PBF((2,0)^k, id) = \frac{1}{k+1}\binom{2k}{k} = C_k$

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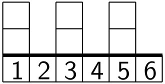

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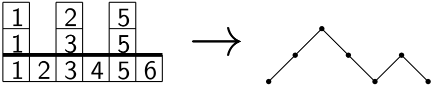
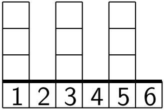


$\gamma = (3,0)^k$



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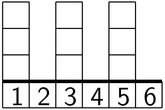

Theorem

$$\#PBF((3,0)^k, id) = \begin{vmatrix} C_k & C_{k+1} \\ C_{k+1} & C_{k+2} \end{vmatrix} = C_{k+2}C_k - C_{k+1}^2$$

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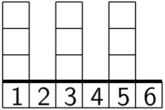

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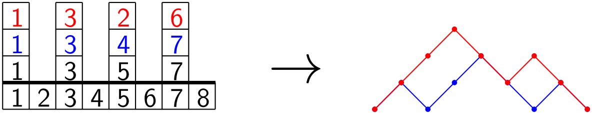
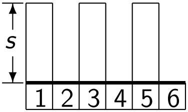


In bijection with pairs of non-crossing Dyck paths.

$\gamma = (s, 0)^k$



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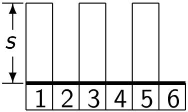

Theorem

If $s \geq 2$, then $\#PBF((s,0)^k, id) = det(A)$ where $A_{i,j} = C_{k+i+j}$ for $i, j \in 0, \ldots, s-2$.

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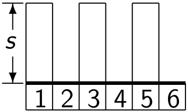

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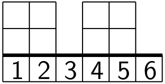
In bijection with $(s-1)$-tuples of non-crossing Dyck paths - also called watermelons.

$\gamma = ((2)^2, 0)^k = (2, 2, 0)^k$



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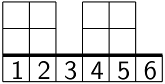

k	1	2	3	4
$\#PBF((2,2,0)^k, id)$	1	3	12	55

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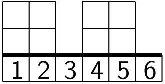

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Theorem

$$\#PBF(((2)^2, 0)^k, id) = \frac{1}{2k+1}\binom{3k}{k}$$

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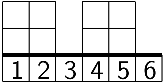

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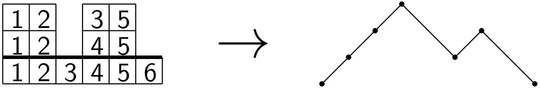
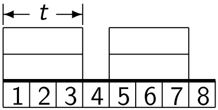


In bijection with double downstep paths.

$\gamma = ((2)^t, 0)^k$



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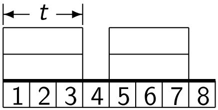

Theorem

$$\#PBF(((2)^t, 0)^k, id) = \frac{1}{tk+1}\binom{(t+1)k}{k}$$

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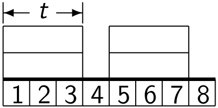

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Rectangles

We know how to build rectangles up (nesting paths) and to the right (steeper downsteps), so we can count any rectangle shape.

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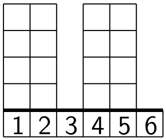
ex.) Let $A_k = \frac{1}{2k+1}\binom{3k}{k}$. Then

$$\#PBF((4,4,0)^k, id) = \begin{vmatrix} A_k & A_{k+1} & A_{k+2} \\ A_{k+1} & A_{k+2} & A_{k+3} \\ A_{k+2} & A_{k+3} & A_{k+4} \end{vmatrix}.$$

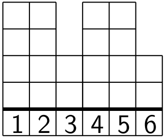


In bijection with 3-tuples of non-crossing double downstep paths.

More Shapes

What if the shape is some repeated pattern, but there are no columns of height 0?

example: $\gamma = (4, 4, 2)^k$



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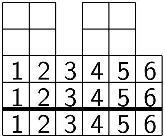

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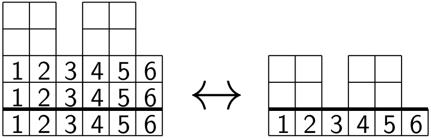

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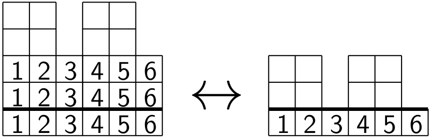
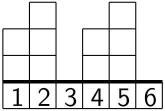


We already know how to count this!

More Shapes

A slightly more interesting shape:



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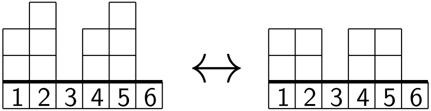

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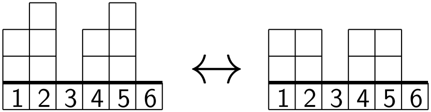
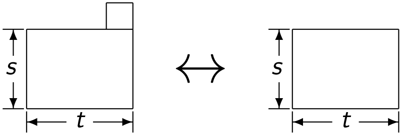


Here, adding an extra cell has no effect of the number of PBFs. It turns out that



Natural Statistics on PBFs

Consider PBFs of shape $(2, 0)^k$ and basement *id*, counted by the Catalan numbers.

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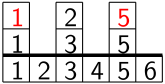

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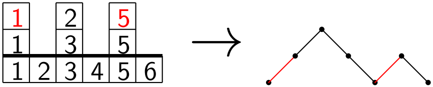
Consider PBFs of shape $(2,0)^k$ and basement *id*, counted by the Catalan numbers.

$$\begin{aligned}\#\text{ constant columns} &= \#\text{ odd numbers in the top row}\\ &= \#\text{ upsteps at odd positions}\end{aligned}$$



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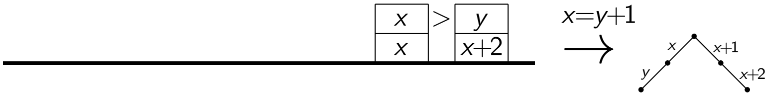

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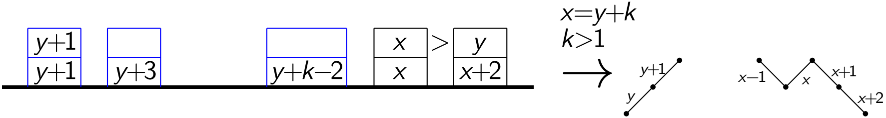

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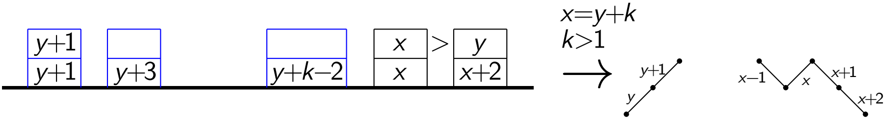

Elements of $\{y+1, \ldots, y+k-2\}$ fill these columns. Moreover, elements of this set cannot appear in any other columns.

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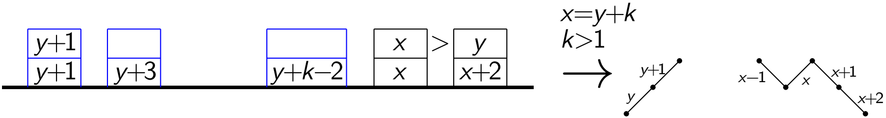

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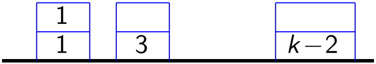

$\longrightarrow$ some Dyck path

some PBF

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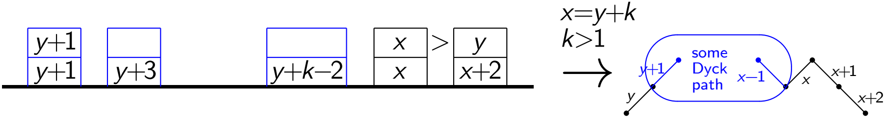

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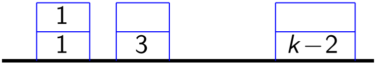

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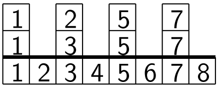
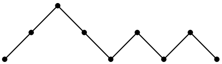


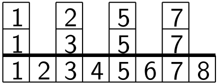


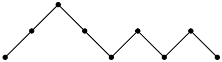
$\rightarrow$ ordered top row elements:

1 2 5 7

Natural Statistics on Paths

$$\begin{aligned}\#\text{ peaks in Dyck path} &= \#\text{ elements } x \text{ in the top row}\\ &\quad\text{such that } x+1 \text{ is not in the top row}\\ &= \#\text{ gaps} + 1\end{aligned}$$





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  - more complicated shapes
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- properties of the insertion algorithm
  - given two PBFs P, Q of shapes γ, δ, reverse the insertion procedure with P as the "recording PBF" and Q as the "insertion PBF"
  - characterize the resulting word