Generating functions for permutations with no consecutive pattern matches within the cycles

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joint work with Jeff Remmel

University of California, San Diego

Permutation Patterns 2012

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1. Introduction to consecutive pattern matches in cycles.

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2. Describe a bijection between derangements and brick tableaux.

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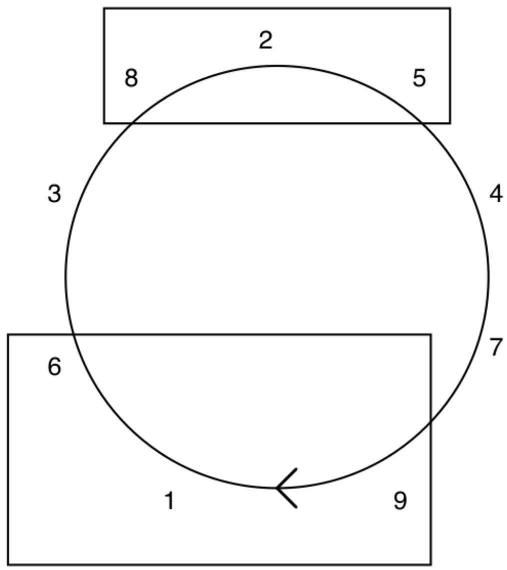
2. Describe a bijection between derangements and brick tableaux.
3. Discuss some generating functions that describe permutations that avoid uncommon sets of patterns.

Consecutive Cycle Pattern Matches



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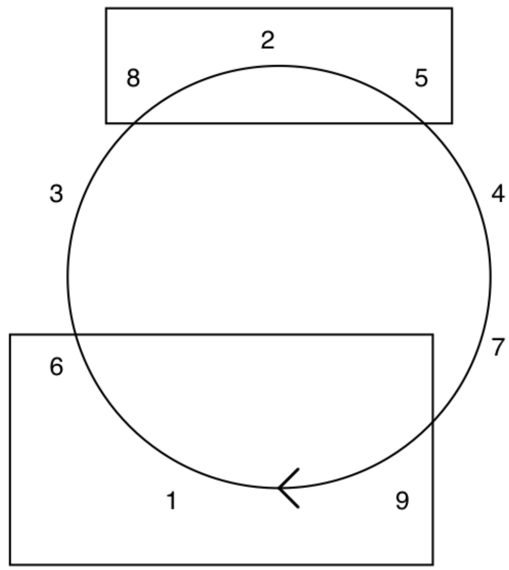

Given a pattern $\tau \in S_j$ and a single cycle $C = (c_0, \ldots, c_{n-1}) \in S_n$, we say that C has a cycle-τ-match at position i if $\text{red}(c_i, \ldots c_{i+j-1}) = \tau$ with the subscripts taken mod n.

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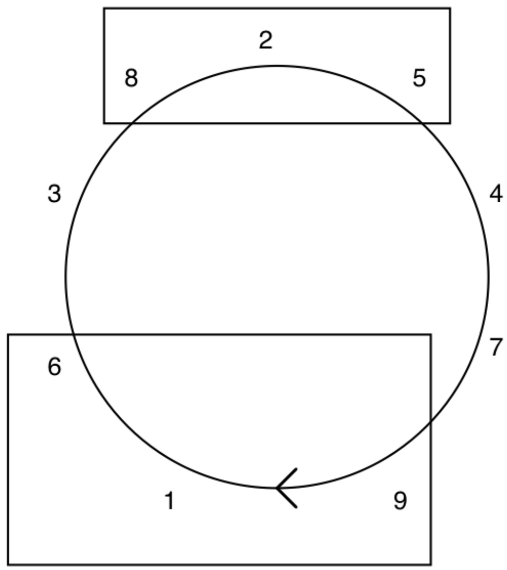

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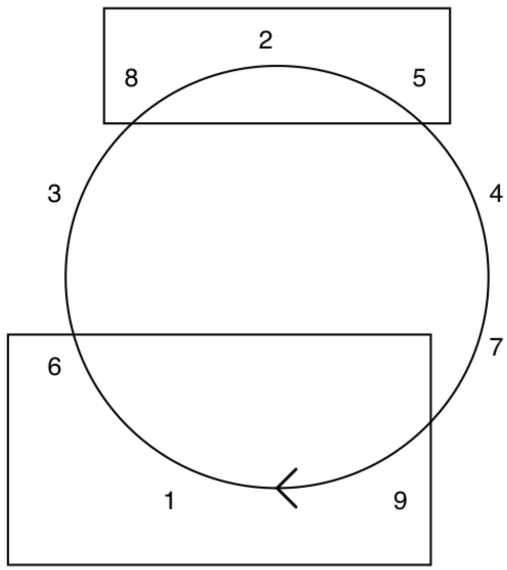

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For a set of patterns $\Upsilon = \{\tau_1, \ldots, \tau_k\}$, $\mathcal{NCM}_n(\Upsilon)$ is the set of all permutations in S_n that have no cycle-τ_i-matches for each $\tau_i \in \Upsilon$.

$$NCM_{\Upsilon}(t) = \sum_{n \geq 0} \frac{t^n}{n!} |\mathcal{NCM}_n(\Upsilon)|$$

$$H(t) = \sum_{n \geq 0} h_n t^n, E(t) = \sum_{n \geq 0} e_n t^n$$

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$$\begin{aligned}
\sum_{n\geq 0} \theta_f(h_n) t^n &= \frac{1}{\theta_f(E(-t))} \\
&= \frac{1}{\sum_{n\geq 0} \theta_f(e_n)(-t)^n} \\
&= \frac{1}{\sum_{n\geq 0} \frac{(-1)^{n-1}}{n!} f(n)(-t)^n} \\
\sum_{n\geq 0} n!\theta_f(h_n)\frac{t^n}{n!} &= \frac{1}{1 - \sum_{n\geq 1} \frac{t^n}{n!} f(n)}
\end{aligned}$$

Combinatorial Interpretation of $n!\theta_f(h_n)$

$$n!\theta_f(h_n) = \sum_{\lambda \vdash n} \binom{n}{b_1 \ldots b_{\ell(\lambda)}} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} f(b_i)$$

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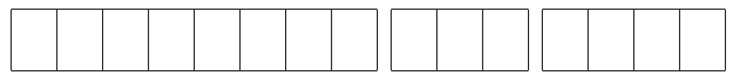

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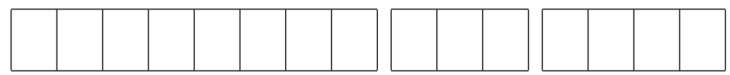
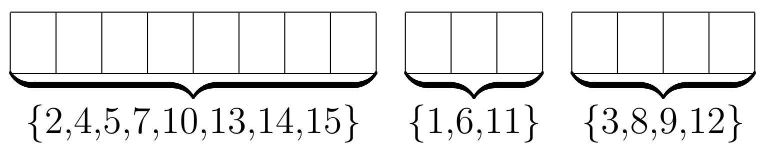


The multinomial coefficient $\binom{n}{b_1 \dots b_{\ell(\lambda)}}$ assigns numbers from $\{1, \dots, n\}$ to each brick.



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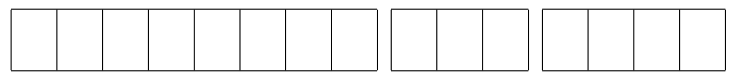

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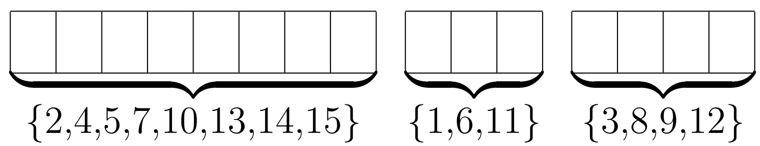
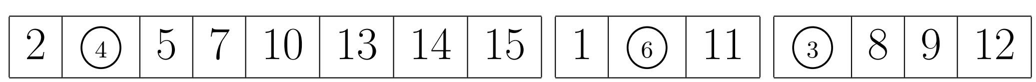


The factors $f(b_i)$ assign a "mark" to each brick to distinguish them from one another.
For example: If $f(n) = n - 1$ then each brick would be marked with a circle on one of the elements that is not the last so that there are $n - 1$ unique bricks filled with the same numbers.



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This is the generating function for derangements!!!!

There must be a bijection.

2	④	5	7	10	13	14	15

1	⑥	11

③	8	9	12

2	④	5	7	10	13	14	15

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(

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$$(4, 15, 14, 13, 10, 7, 5, 2)(6, 11, 1, 3, 12, 9, 8)$$

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2 3 ⑧ 12	④ 7	1 5 ⑥ 9 10 11

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This is the generating function for single cycles of length ≥ 2.
What are the preimages of single cycles under our bijection Φ_n?

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$$\log\left(\frac{1}{1 - \sum_{n\geq 1} \frac{t^n}{n!} f(n)}\right)$$

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The objects described by $n!\theta_{f_1}(h_n)$ are the circled brick tableaux described earlier with the restriction that there will be no brick of size 2.

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The objects described by $n!\theta_{f_1}(h_n)$ are the circled brick tableaux described earlier with the restriction that there will be no brick of size 2.

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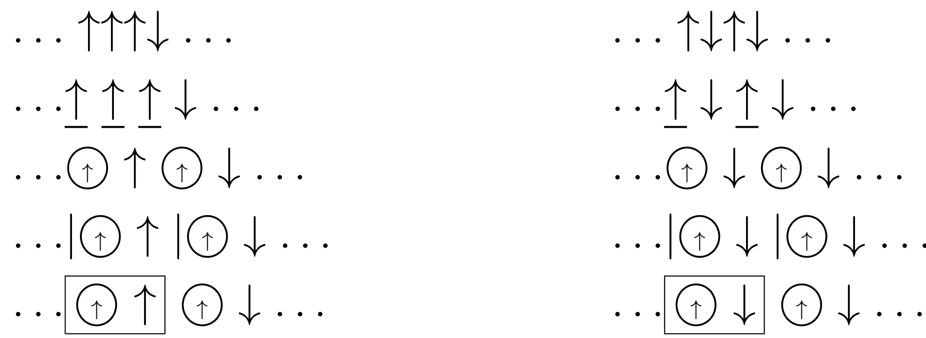

So the generating function

$$\log\left(\frac{1}{1-\sum_{n\geq 3}\frac{t^n}{n!}(n-1)}\right)$$

counts the set of all cycles with length greater than 3 that have no cycle-τ-match for any pattern τ in the set

$$\begin{aligned}\Upsilon_1 =&\{\uparrow\uparrow\uparrow, \uparrow\downarrow\uparrow\downarrow\}\\ =&\{1234, 13254, 14253, 14352, 15243, 15342, 23154, 24153, 24351, 25143, 25341,\\ &34152, 34251, 35142, 35241, 45132, 45231\}\end{aligned}$$

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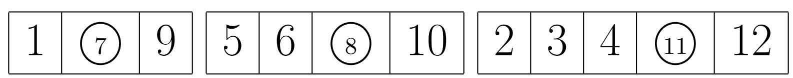

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1	$\textcircled{7}$	9 ‖	5	6	$\textcircled{8}$	10 ‖	2	3	4	$\textcircled{11}$	12

$\textcircled{7}$	9	1 ‖	$\textcircled{8}$	10	6	5 ‖	$\textcircled{11}$	12	4	3	2

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1 $\textcircled{7}$ 9 | 5 6 $\textcircled{8}$ 10 | 2 3 4 $\textcircled{11}$ 12

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$$(7, 9, 1, 8, 10, 6, 5, 11, 12, 4, 3, 2)$$

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Notice that whenever there is a peak, it must be a 231 match.

So the generating function

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1	③	7	9	10

④	5	8

2	6	⑪	12	13

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[1 | ③ | 7 | 9 | 10] [④ | 5 | 8] [2 | 6 | ⑪ | 12 | 13]

[③ | 10 | 9 | 7 | 1] [④ | 8 | 5] [⑪ | 13 | 12 | 6 | 2]

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1	$\textcircled{3}$	7	9	10		$\textcircled{4}$	5	8		2	6	$\textcircled{11}$	12	13

$\textcircled{3}$	10	9	7	1		$\textcircled{4}$	8	5		$\textcircled{11}$	13	12	6	2

$$(3, 10, 9, 7, 1, 4, 8, 5, 11, 13, 12, 6, 2)$$

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③	10	9	7	1	④	8	5	⑪	13	12	6	2

Bricks: [1, ③, 7, 9, 10] [④, 5, 8] [2, 6, ⑪, 12, 13] and [③, 10, 9, 7, 1] [④, 8, 5] [⑪, 13, 12, 6, 2]

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Notice that whenever there is a peak, it must be a 132 match.

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Further work: What other kinds of objects can we describe with different functions $f(n)$?

$f(n) = n - 1$	derangements
$f(n) = \begin{cases} 0 & \text{if } n = 1, 2 \\ n - 1 & \text{if } n \geq 3 \end{cases}$	$\{\uparrow\uparrow\uparrow, \uparrow\downarrow\uparrow\downarrow\}$
$f(n) = \begin{cases} 0 & \text{if } n = 1, 2, 3 \\ n - 1 & \text{if } n \geq 4 \end{cases}$	$\{\uparrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow\uparrow\downarrow\}$
$f(n) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4 \\ n - 1 & \text{if } n \geq 5 \end{cases}$	$\{\uparrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow\uparrow, \uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\}$
$f(n) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4 \\ 1 & \text{if } n \geq 5 \end{cases}$	$\{\uparrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow\uparrow, \uparrow\downarrow\downarrow\downarrow\uparrow\downarrow, 132\}$
$f(n) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4 \\ n - 2 & \text{if } n \geq 5 \end{cases}$	$\{\uparrow\uparrow\uparrow, \uparrow\downarrow\uparrow, \uparrow\downarrow\downarrow\uparrow, \uparrow\downarrow\downarrow\downarrow\uparrow\downarrow, 231\}$
$f(n) = 2n - 1$	signed derangements

$$\frac{1}{1-\sum_{n\geq 1}\frac{t^n}{n!}x(n-1)} = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{T\in\mathcal{F}_{D,n}} x^{\#\text{ of bricks of }T}$$
$$= \frac{1}{1+(1-t)xe^t - x}$$

$$f(n) = \begin{cases} 0 & \text{if } n = 1,2 \\ x(n-1) & \text{if } n \geq 3 \end{cases}$$

$$f(n) = \begin{cases} 0 & \text{if } n = 1,2 \\ x(1) & \text{if } n \geq 3 \end{cases}$$

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