Stanley–Wilf Limits of Layered Patterns

Permutation Patterns 2012

Anders Claesson, Vít Jelínek, Einar Steingrímsson

Stanley–Wilf Limits

Definition

$\mathrm{Av}(\pi)$ is the set of π-avoiding permutations.
$\mathrm{Av}_n(\pi)$ is the set of π-avoiding permutations of size n.

The Stanley–Wilf limit of π, denoted by $L(\pi)$, is defined as

$$L(\pi) := \lim_{n\to\infty} \sqrt[n]{|\mathrm{Av}_n(\pi)|}.$$

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Direct Sums

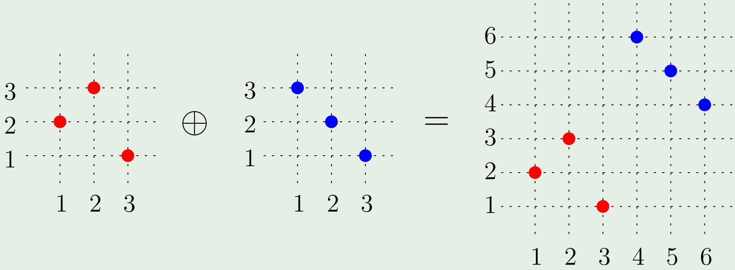
Definition

Given two permutations $\pi = \pi_1, \ldots, \pi_k$ and $\sigma = \sigma_1, \ldots, \sigma_m$, define the direct sum $\pi \oplus \sigma$ as

$$\pi \oplus \sigma = \pi_1, \ldots, \pi_k, \sigma_1 + k, \ldots, \sigma_m + k.$$

Example

$$231 \oplus 321 = 231654$$



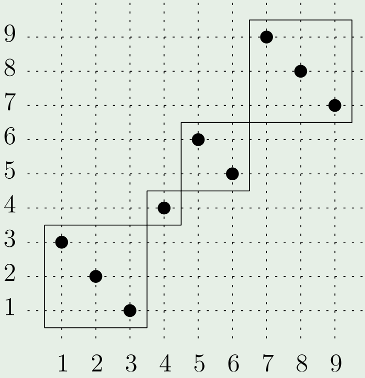
Layered Permutations

Definition

A layered permutation is a direct sum of decreasing permutations.

Example

$\pi = 321465987 = 321 \oplus 1 \oplus 21 \oplus 321$ is a layered permutation



Some Results on Stanley–Wilf Limits

For a general permutation π of size k:

- $\Omega(k^2) \leq L(\pi) \leq 2^{O(k \log k)}$

For layered π:

- $(k-1)^2 \leq L(\pi) \leq 2^{O(k)}$

For specific patterns:

- $L(123) = L(132) = 4$
- $L(123 \cdots k) = (k-1)^2$
- $L(1342) = L(2413) = 8$
- $9.47 \leq L(1324) \leq 288$

Conjecture: Among all the patterns π of a given size, $L(\pi)$ is maximized by a layered pattern.

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Merging

Definition

Permutation π is a merge of permutations σ and τ if the symbols of π can be colored red and blue, so that the red symbols are order-isomorphic to σ and the blue ones to τ.

Example

3175624 is a merge of 231 and 1342.

Definition

For two sets P and Q of permutations, let $\textsc{Merge}[P, Q]$ be the set of permutations obtained by merging a $\sigma \in P$ with a $\tau \in Q$.

Lemma (Albert et al., Bóna)

If $Av(\pi) \subseteq \textsc{Merge}[Av(\sigma), Av(\tau)]$, then

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The Key Lemma

Lemma

For any patterns α, β *and* γ *we have*

$$Av(\alpha \oplus \beta \oplus \gamma) \subseteq \textsc{Merge}[Av(\alpha \oplus \beta), Av(\beta \oplus \gamma)],$$

and therefore $\sqrt{L(\alpha \oplus \beta \oplus \gamma)} \leq \sqrt{L(\alpha \oplus \beta)} + \sqrt{L(\beta \oplus \gamma)}$.

Remark

The special case $\beta = 1$ has been proved by Bóna, who actually shows $\sqrt{L(\alpha \oplus 1 \oplus \gamma)} = \sqrt{L(\alpha \oplus 1)} + \sqrt{L(1 \oplus \gamma)}$.

Example

Taking $\alpha = 1$, $\beta = 21$, and $\gamma = 1$ gives

$$\sqrt{L(1324)} \leq \sqrt{L(132)} + \sqrt{L(213)} = 4,$$

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Example

Define $\lambda_k := k(k-1)\cdots 1$. Consider $\pi = \lambda_3 \oplus \lambda_1 \oplus \lambda_7 \oplus \lambda_6 \oplus \lambda_2$.

$$\begin{aligned}\sqrt{L(\pi)} &\leq \sqrt{L(\lambda_3 \oplus \lambda_1)} + \sqrt{L(\lambda_1 \oplus \lambda_7 \oplus \lambda_6 \oplus \lambda_2)} \\ &\leq \sqrt{L(\lambda_3 \oplus \lambda_1)} + \sqrt{L(\lambda_1 \oplus \lambda_7)} + \sqrt{L(\lambda_7 \oplus \lambda_6 \oplus \lambda_2)} \\ &\leq \sqrt{L(\lambda_3 \oplus \lambda_1)} + \sqrt{L(\lambda_1 \oplus \lambda_7)} + \sqrt{L(\lambda_7 \oplus \lambda_6)} + \sqrt{L(\lambda_6 \oplus \lambda_2)} \\ &= 3 + 7 + 12 + 7 = 29\end{aligned}$$

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Corollary

Let π *be a layered pattern of size* k *with* $m \geq 2$ *layers of lengths* $k_1, k_2, \ldots, k_m$. *Then*

$$L(\pi) \leq (2k - k_1 - k_m - m + 1)^2 .$$

In particular, $L(\pi) < 4k^2$.

Proof of The Key Lemma

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Proof:

- Fix α, β and γ as in the lemma.
- Choose $\pi = (\pi_1, \ldots, \pi_n) \in \text{Av}(\alpha \oplus \beta \oplus \gamma)$.
- Goal: color elements of π red and blue, so that the red part avoids $\alpha \oplus \beta$ and the blue part avoids $\beta \oplus \gamma$.
- The trick: color elements $\pi_1, \ldots, \pi_n$ left to right. An element π_i is colored blue if and only if one of the following holds:
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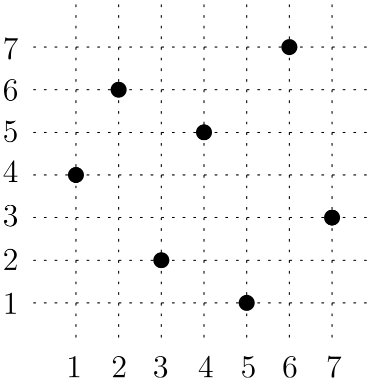
Example

Take $\alpha = 1$, $\beta = 21$, $\gamma = 12$, and $\pi = 4725163 \in \mathrm{Av}(13245)$.
Goal: show that $\pi \in \textsc{Merge}[\mathrm{Av}(132), \mathrm{Av}(2134)]$



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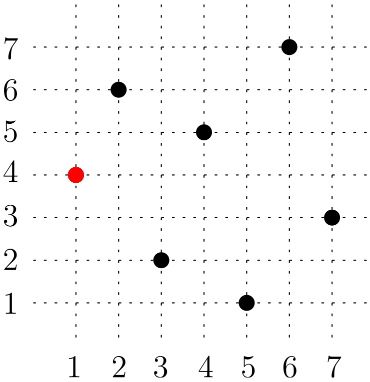

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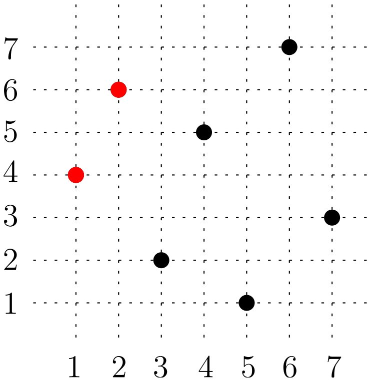

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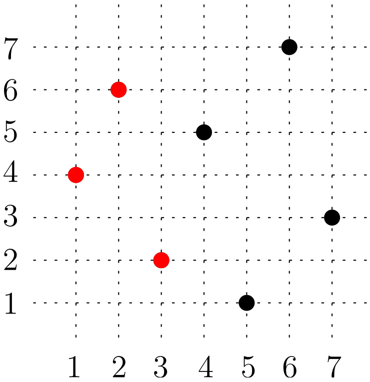

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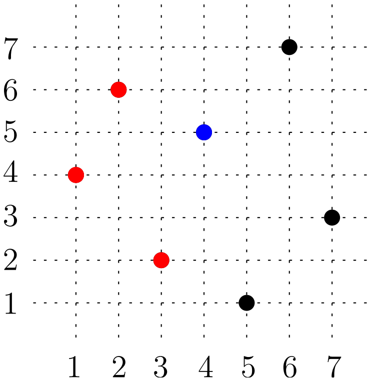

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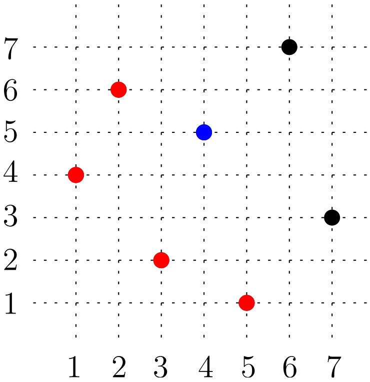

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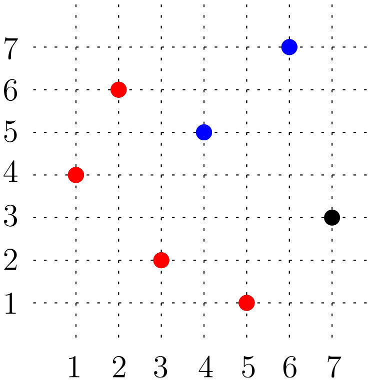

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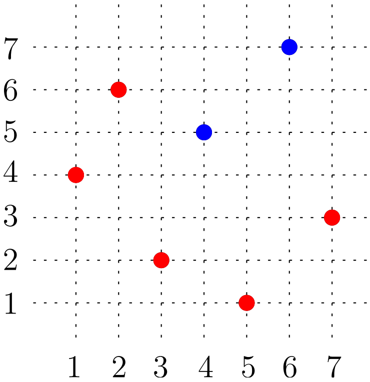

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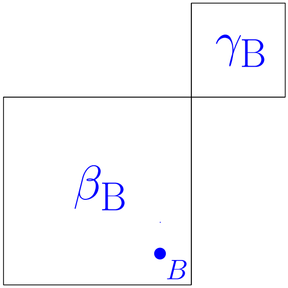

$B :=$ bottommost point of β_B

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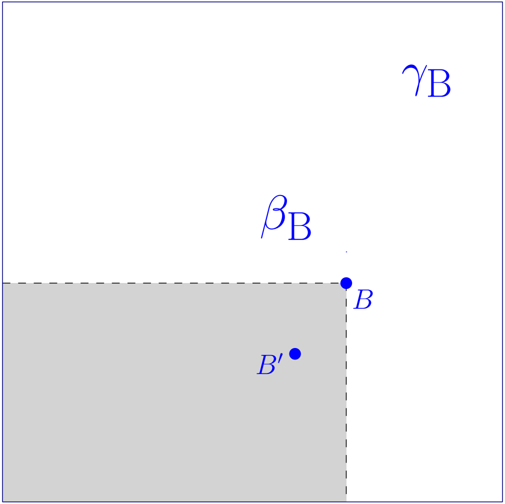

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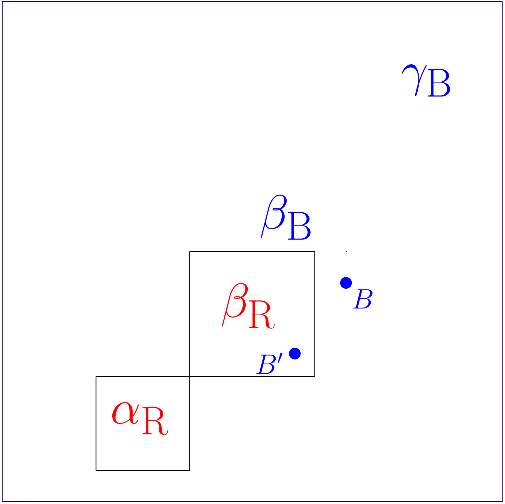

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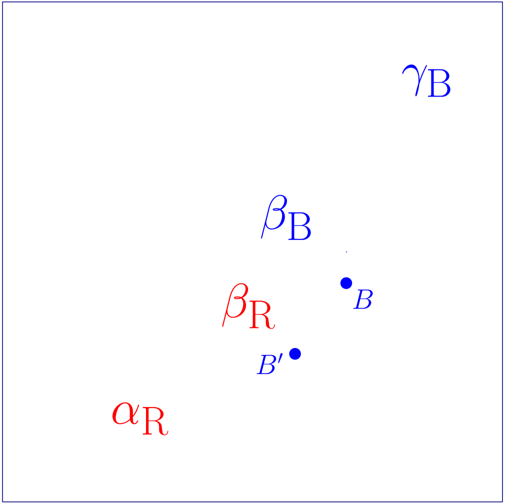

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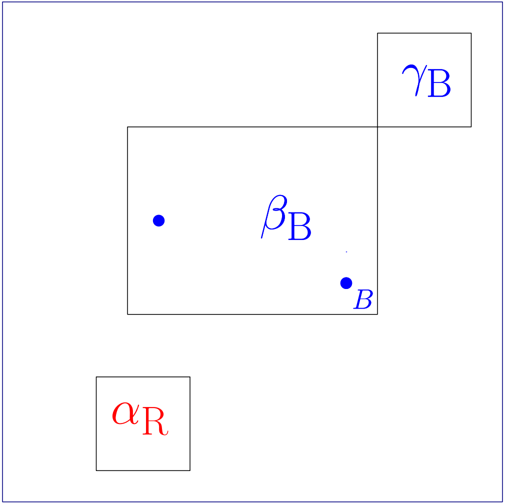

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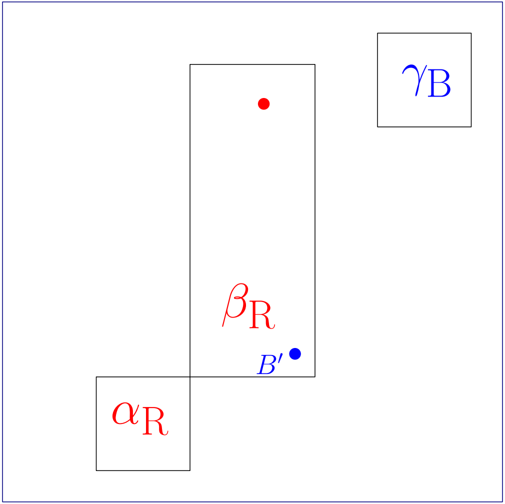

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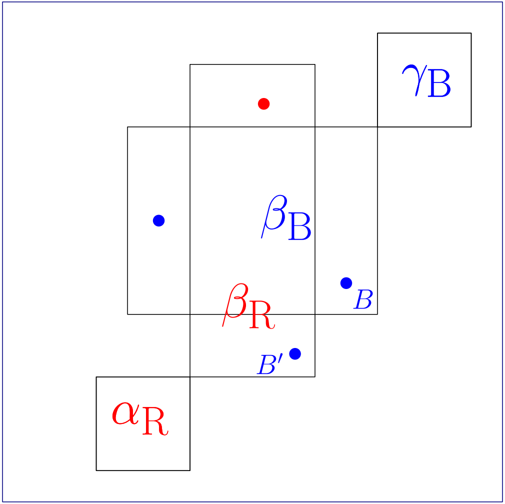

Remarks and Open Problems

- Let $\mathrm{Av}_n^m(1324)$ be the set of 1324-avoiding permutations of size n with m inversions. Conjecture:

$$\forall m \forall n\colon |\mathrm{Av}_n^m(1324)| \leq |\mathrm{Av}_{n+1}^m(1324)|$$

- If the conjecture holds, then $L(1324) \leq e^{\pi\sqrt{2/3}} \simeq 13.002$.
- For what other patterns π, σ, τ do we have $\mathrm{Av}(\pi) \subseteq \textsc{Merge}[\mathrm{Av}(\sigma), \mathrm{Av}(\tau)]$?
- For what pattern π of size k is the value $L(\pi)$ maximized (or minimized)?

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The End

Thank you for your attention!