Expected Patterns in Permutations Avoiding 123

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June 12, 2012

Introduction

Example

The permutation 3 1 5 2 4 contains the pattern 3 1 2 twice (the first, second and forth entries and the third, forth and fifth).

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The set $\{2341, 1234, 4321\}$ contains the pattern 123 exactly 5 times.

Data

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Av 123

length	123	132	213	231	312	321
3	0	1	1	1	1	1

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length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16

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length	123	132	213	231	312	321
3	0	1	1	1	1	1
4	0	9	9	11	11	16
5	0	57	57	81	81	144
6	0	312	312	500	500	1016
7	0	1578	1578	2794	2794	6271

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Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13

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Av 132

length	123	132	213	231	312	321
3	1	0	1	1	1	1
4	10	0	11	11	11	13
5	68	0	81	81	81	109
6	392	0	500	500	500	748
7	2063	0	2794	2794	2794	4570

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Definition

Similarly, for a pattern q and a set $\mathcal{C}$ of permutations, define

$$\mathrm{f}_q(\mathcal{C}) = \sum_{p \in \mathcal{C}} \mathrm{f}_q(p).$$

Preliminaries

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Fact

$$(\mathsf{f}_{12} + \mathsf{f}_{21})(\mathrm{Av}_n\, 123) = \binom{n}{2} c_n.$$

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Theorem (Cheng, Eu, Fu 2007)

$$\mathrm{f}_{12}(\mathrm{Av}_n\, 123) = \sum_{k=1}^{n-1} c_k 4^{n-k-1} = 4^{n-1} - \binom{2n-1}{n}$$

$$\sum_{n \geq 0} \mathrm{f}_{12}(\mathrm{Av}_n\, 123) x^n = \frac{1 - 2x - \sqrt{1-4x}}{2(1-4x)}.$$

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$$(4\,\mathrm{f}_{132} + 2\,\mathrm{f}_{231})(\mathrm{Av}_n\,123) = (n-2)\,\mathrm{f}_{12}(\mathrm{Av}_n\,123).$$

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$$(4\,\mathrm{f}_{132} + 2\,\mathrm{f}_{231})(\mathrm{Av}_n\,123) = (n-2)\,\mathrm{f}_{12}(\mathrm{Av}_n\,123).$$

Proof.

Rewrite as

$$(n-2)\,\mathrm{f}_{12} - \mathrm{f}_{132} - \mathrm{f}_{213} = \mathrm{f}_{231} + \mathrm{f}_{312} + \mathrm{f}_{132} + \mathrm{f}_{213}\,.$$

Both sides count the number of length three patterns with at least one non-inversion. □

Lemma

Let $a_n = \mathrm{f}_{132}(\mathrm{Av}_n\, 123)$, $b_n = \mathrm{f}_{231}(\mathrm{Av}_n\, 123)$, $d_n = \mathrm{f}_{321}(\mathrm{Av}_n\, 123)$, *and* $j_n = \mathrm{f}_{12}(\mathrm{Av}_n\, 123)$. *Let* $A(x), B(x), D(x), J(x)$ *be their respective generating functions. Then*

$$\begin{array}{lllcl} 2A(x) & +2B(x) & +D(x) & = & \frac{x^3}{6}(C(x))''' \\ 4A(x) & +2B(x) & & = & x^3(J(x)/x^2)' \end{array}$$

Indecomposable Permutations

Definition

We say that a permutation $p = p_1 p_2 \ldots p_n$ is *decomposable* if there exists an integer k so that each of the entries $p_1, \ldots p_k$ is greater than east of the entries $p_{k+1}, \ldots p_n$. Otherwise, we say that p is *indecomposable*

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Definition

Denote by $\mathrm{Av}_n^* 123$ the set of indecomposable n-permutations which avoid 123.

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$|\operatorname{Av}_n^* 123| = c_{n-1}$.

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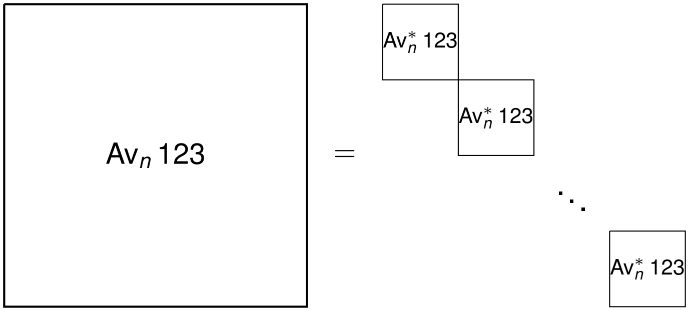

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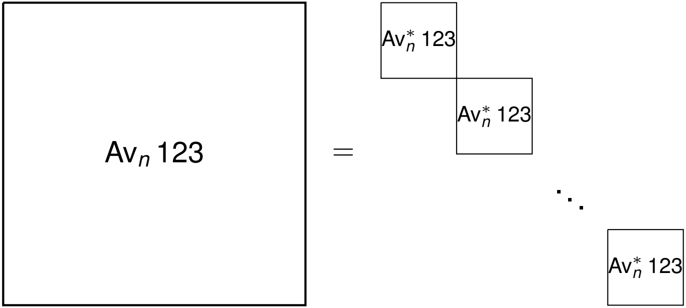

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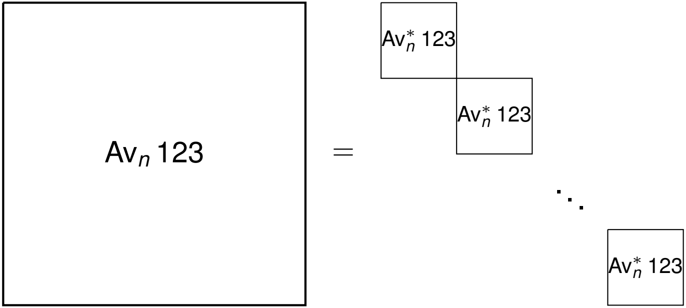

$$C(x) = 1 + C^*(x) + C^*(x)^2 + \ldots = \frac{1}{1 - C^*(x)}$$

$$C^*(x) = \frac{C(x) - 1}{C(x)} = xC(x).$$

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Indecomposable Permutations

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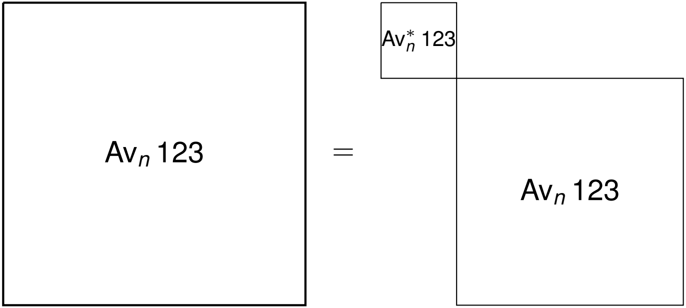
Alternate Proof.



$$C(x) = C^*(x)C(x) + 1$$

$$C^*(x) = \frac{C(x) - 1}{C(x)} = xC(x).$$

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$$\begin{array}{cccc} 2A(x) & +2B(x) & +D(x) & = & \frac{x^3}{6}(C(x))''' \\ 4A(x) & +2B(x) & & = & x^3(J(x)/x^2)' \\ 2A^*(x) & +2B^*(x) & +D^*(x) & = & \frac{x^3}{6}(xC(x))''' \\ 4A^*(x) & +2B^*(x) & & = & x^3(J^*(x)/x^2)' \end{array}$$

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Corollary

$$C(x)A(x) = xJ(x)C'(x)$$

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$$A(x) + B(x) = 2\sum_{n\geq 0} \Big(\mathrm{f}_{213}\left(\mathrm{Av}^*_n\, 132\right) + \mathrm{f}_{231}\left(\mathrm{Av}^*_n\, 132\right)\Big) x^n$$

$$A(x) + B(x) = xB^*(x)$$

$$J^*(x) = 2A^*(x)$$

Lemma

$$A^*(x) = \frac{x^3 C(x)}{(1-4x)^{3/2}}$$

One Last Definition

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Definition

A *Dyck path* of length $2n$ (or of semilength n) is a path in the plane from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ and $(1,-1)$ which never crosses below the x-axis.

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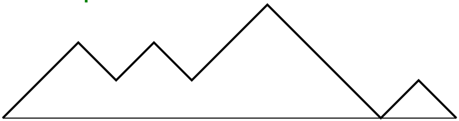

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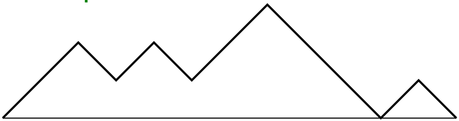 $= UUDUDUUDDDUD.$

Fact

There are exactly c_n Dyck paths of semilength n.

Sketch of proof

Let $p = 4\ 3\ 7\ 6\ 1\ 5\ 2$, and count 213 patterns.

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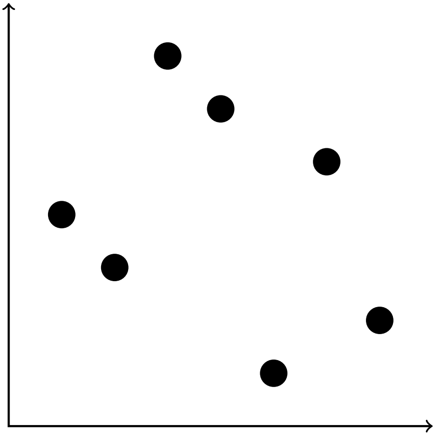

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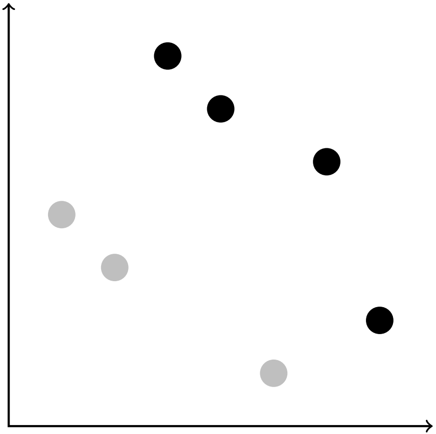

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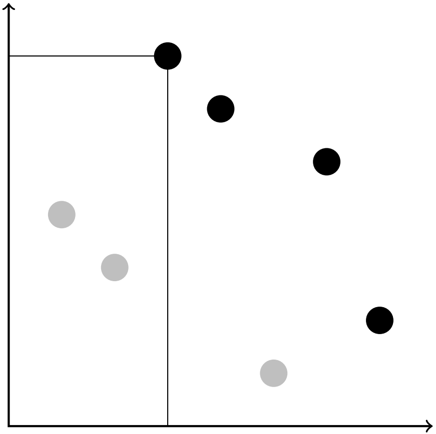

$$f_{213}(p) = \binom{2}{2}$$

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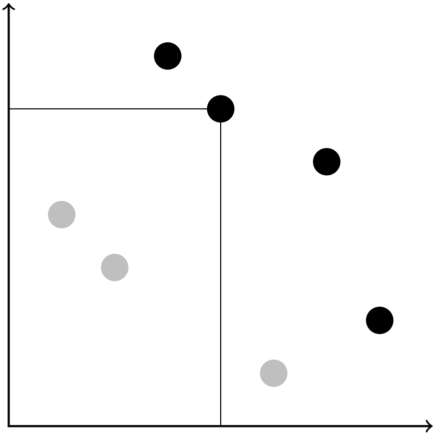

$$f_{213}(p) = \binom{2}{2} + \binom{2}{2}$$

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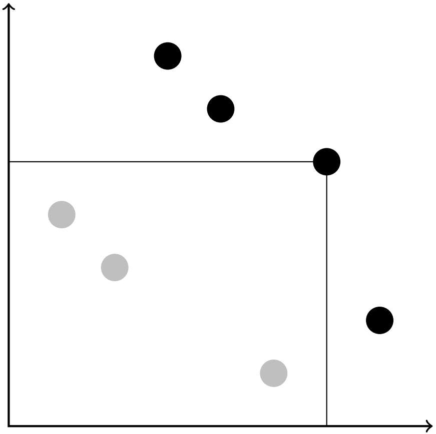

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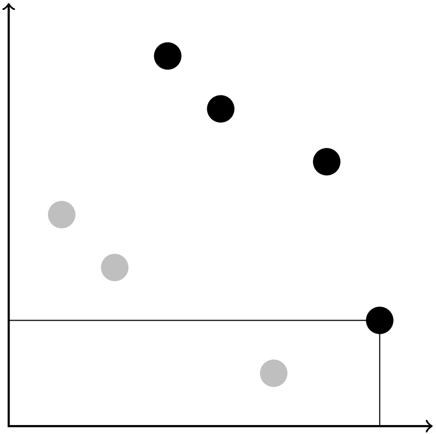

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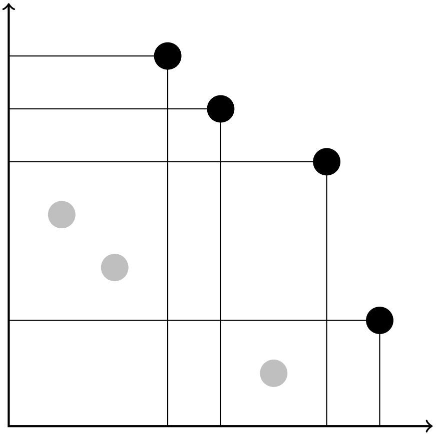

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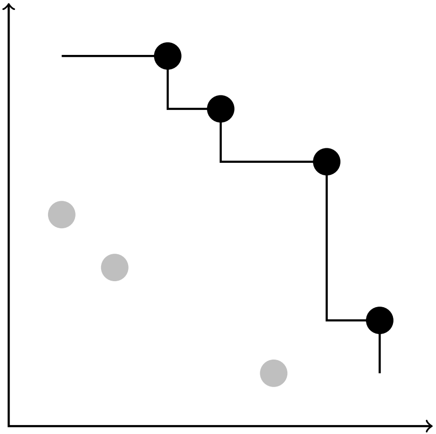

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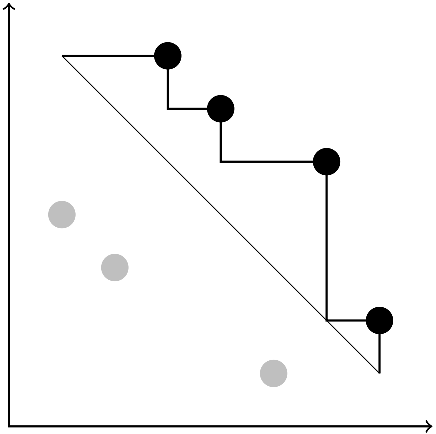

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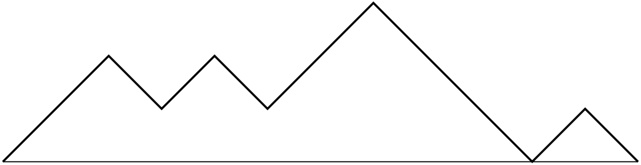

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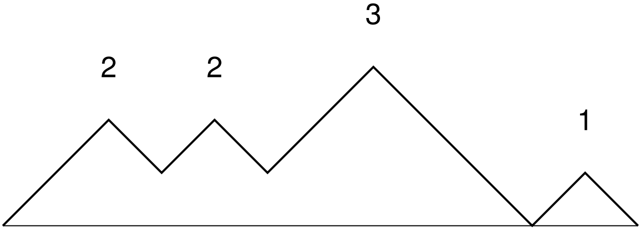

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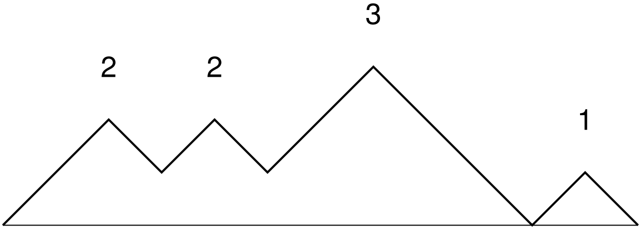
Sketch of proof

Let $h_{n,k}$ denote the total number of peaks at height k in all Dyck paths of semilength n. Let $H(x,u) = \sum_{n,k\geq 0} h_{n,k} x^n u^k$.



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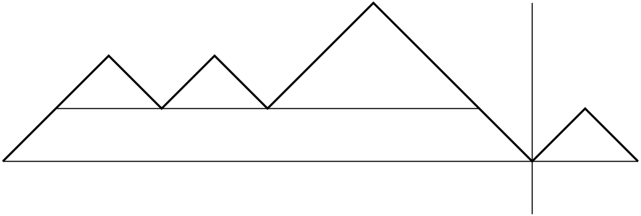

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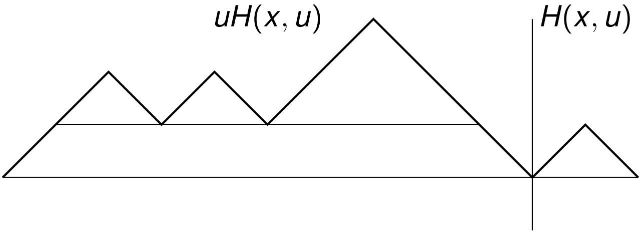

$$H(x,u) = ux(H(x,u)+1)C(x) + xC(x)H(x,u)$$

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Corollaries

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Corollary

$$\mathrm{f}_{231}(\mathrm{Av}_n\,123) = \mathrm{f}_{231}(\mathrm{Av}_n\,132)$$

Corollaries

$$A(x) = \frac{x-1}{2(1-4x)} - \frac{3x-1}{2(1-4x)^{3/2}}$$

$$B(x) = \frac{3x-1}{(1-4x)^2} - \frac{4x^2-5x+1}{(1-4x)^{5/2}}$$

$$D(x) = \frac{8x^3-20x^2+8x-1}{(1-4x)^2} - \frac{36x^3-34x^2+10x-1}{(1-4x)^{5/2}}$$

Corollaries

$$a_n = \frac{n+2}{4}\binom{2n}{n} - 3 \cdot 2^{2n-3}$$

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$$\begin{aligned} d_n = {} & \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} \\ & + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2} - 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1) \cdot 4^{n-1}. \end{aligned}$$

Corollaries

$$a_n \sim \sqrt{\frac{n}{\pi}} 4^n$$

$$b_n \sim \frac{n}{2} 4^n$$

$$d_n \sim \frac{8}{3} \sqrt{\frac{n^3}{\pi}} 4^n.$$

Larger patterns

Larger patterns

Lemma

$$\begin{array}{llll} 2A(x) & +2B(x) & +D(x) & = \frac{x^3}{6}(C(x))''' \\ 4A(x) & +2B(x) & & = x^3(J(x)/x^2)' \end{array}$$

Larger patterns

Lemma

$$
\begin{aligned}
2A(x) + 2B(x) + D(x) &= \frac{x^3}{6}(C(x))''' \\
4A(x) + 2B(x) &= x^3(J(x)/x^2)'
\end{aligned}
$$

Proposition

For large enough n, the descending pattern of length k occurs more often than any other length k pattern in $\mathrm{Av}_n\,123$.

Thanks for listening!