The Möbius function of the consecutive pattern poset

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Given a locally finite poset P, its *Möbius function* is defined as

$$\begin{aligned}\mu :& P \times P \longrightarrow \mathbf{C} \\ :&(x,x) \longmapsto 1, \\ :&(x,y) \longmapsto -\sum_{x \leq z < y} \mu(x,z) = -\sum_{x < z \leq y} \mu(z,y) \qquad (x \neq y)\end{aligned}$$

Note that $\mu(x,y) = 0$ if $x \not\leq y$.

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Given permutations σ, τ, we say that $\sigma = a_1 \cdots a_k$ occurs as a *pattern* in $\tau = b_1 \cdots b_n$ whenever there exists a set of k elements of τ, say $\{b_{i_1}, \cdots b_{i_k}\}$, appearing in τ in the above order, whose elements appear in the same order of size as the elements of σ.

Example. $\sigma = 231$ appears 8 times in $\tau = 253641$.

The problem of the computation of the Möbius function in the pattern poset (posed by Wilf) is still open in its full generality.
Partial results are due to Sagan and Vatter (2006), Steingrímsson and Tenner (2010) and Burstein, Jelínek, Jelínkova and Steingrímsson (2011).

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A permutation σ occurs in a permutation τ as a *consecutive pattern* when there is an occurrence of σ in τ made of consecutive elements of τ.

We set $\sigma \leq \tau$ when σ occurs in τ as a consecutive pattern, and denote with $\mathcal{P}$ the resulting poset.

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Our problem

Given $\sigma, \tau \in \mathcal{P}$ such that $\sigma \leq \tau$, we wish to compute

$$\mu(\sigma, \tau).$$

The computation of $\mu(\sigma, \tau)$ depends on how many occurrences of σ appear in τ.

- If τ contains a single occurrence of σ, $\mu(\sigma, \tau)$ is easily worked out.
- If τ contains two or more occurrences of σ, we provide a recursive procedure to find $\mu(\sigma, \tau)$.

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A useful fact

Observe that, given $\tau \in \mathcal{P}$, it is not difficult to understand the set of elements of $\mathcal{P}$ which are covered by τ.

In fact, "going down" in $\mathcal{P}$ essentially means deleting one element from one tail of τ (and then renaming the remaining ones to get a permutation). Thus

- if τ is a monotone permutation, then it covers only one element (which is in turn a monotone permutation);
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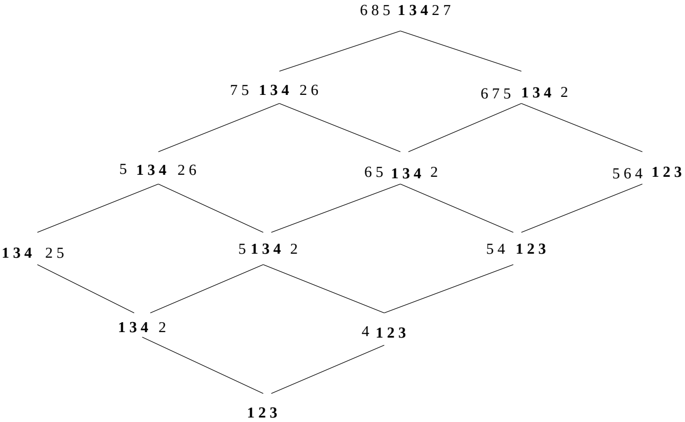

The case of one occurrence

Theorem

Suppose σ occurs precisely once in τ, and that τ has tails of lengths a and b, respectively. Then $\mu(\sigma, \tau) = 1$ if $a = b = 0$ or $a = b = 1$, and $\mu(\sigma, \tau) = -1$ if $a = 0, b = 1$, or $a = 1, b = 0$. Otherwise, $\mu(\sigma, \tau)$ is 0.

Small intervals

Lemma

Suppose σ occurs in τ and that $r = |\tau| - |\sigma| \leq 2$. Then, if $r = 0$, $\mu(\sigma, \tau) = 1$. If $r = 1$, we have $\mu(\sigma, \tau) = -1$. If $r = 2$, then $\mu(\sigma, \tau) = 0$ if τ (and thus also σ) is a monotone permutation, otherwise $\mu(\sigma, \tau) = 1$.

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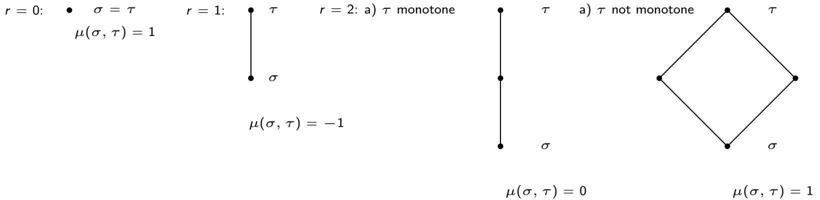

Interior of a permutation

Given τ, define:

- ${}'\tau$: obtained from τ by removing the first letter
- τ': obtained from τ by removing the last letter
- ${}'\tau'$: obtained from τ by removing the first and the last letter

${}'\tau'$ will be called the *interior* of τ.

Example. $\tau = 68513427 \rightsquigarrow {}'\tau = 7513426$, $\tau' = 6751342$ and ${}'\tau' = 651342$.

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Permutations without interior

It τ has no interior in $[\sigma, \tau]$, then it is easy to compute $\mu(\sigma, \tau)$.

Lemma

Suppose σ occurs at least twice in τ and that $|\tau| - |\sigma| \geq 3$. If 'τ' does not lie in $[\sigma, \tau]$, we have $\mu(\sigma, \tau) = 1$.

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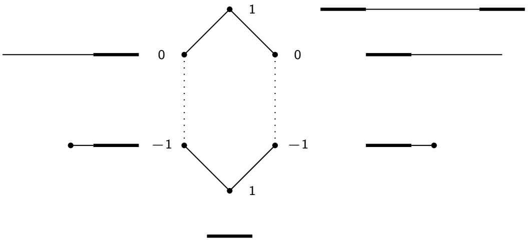

Prefixes, suffixes, bifixes...

From now on, we will suppose that

- $|\tau| - |\sigma| \geq 3$
- $'\tau' \in [\sigma, \tau]$

To deal with the remaining cases, we need some definitions.

Given a permutation τ, its *prefix* (resp. *suffix*) *pattern of length* k is the permutation of length k order isomorphic to the prefix (resp. suffix) of τ of length k. In case the prefix and suffix patterns of length k of τ coincide, we say that τ has a *bifix pattern of length* k.

Example. $\pi = 68513427$;

- 231 is the prefix pattern of π of length 3
- 213 is the suffix pattern of π of length 3

$\pi = 431825976$ has a bifix pattern of length 3, which is 321.

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A crucial lemma

Lemma

Given σ and τ in $\mathcal{P}$, let

$$'C = \{\rho \in [\sigma, \tau] \mid \rho < {'\tau}, \rho \not\leq {'\tau'}\}$$

and let

$$C' = \{\rho \in [\sigma, \tau] \mid \rho < \tau', \rho \not\leq {'\tau'}\}.$$

The sets $'C$ and C' are chains, and $'C \cap C'$ has at most one element. Moreover, if $z \in C' \setminus {'C}$ or $z \in {'C} \setminus C'$ then $[z, \tau]$ is a chain.

Proof.

- $\rho \in {}'C \Rightarrow \rho \not\leq {}'\tau' \Rightarrow \rho$ is a suffix pattern of $\tau \Rightarrow {}'C$ is a chain
- $\rho \in C' \Rightarrow \rho \not\leq {}'\tau' \Rightarrow \rho$ is a prefix pattern of $\tau \Rightarrow C'$ is a chain

$y \leq {}'\tau'$, a contradiction.

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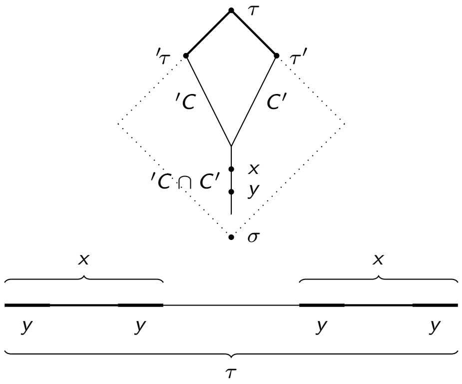

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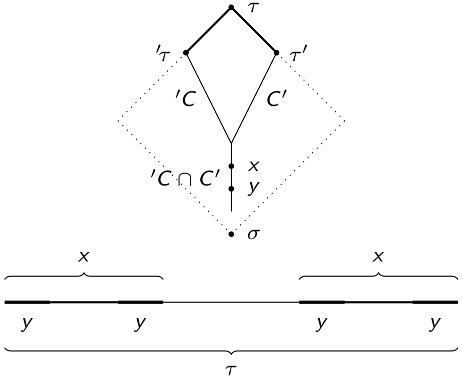

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The carrier element

Given $\sigma \leq \tau$, if $'C \cap C'$ is nonempty, let $\{\mathcal{C}\} = {'C} \cap C'$. $\mathcal{C}$ is called the *carrier element* of $[\sigma, \tau]$.

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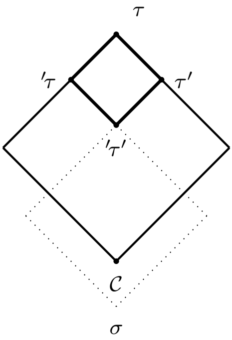

The main recursion

Theorem

Suppose τ has at least two occurrences of σ and that $|\tau| - |\sigma| \geq 3$. Assume that $'\tau'$ lies in $[\sigma, \tau]$. Then, if $[\sigma, \tau]$ has no carrier element, we have $\mu(\sigma, \tau) = 0$. Otherwise, $\mu(\sigma, \tau) = \mu(\sigma, \mathcal{C})$.

Proof. We compute $\mu(\sigma, \tau)$ from top to bottom.

$$z \in (C' \setminus {}'C) \cup ({}'C \setminus C') \Rightarrow \mu(z, \tau) = 0.$$

If $\mathcal{C}$ doesn't exist, then $y < {}'\tau' \Rightarrow \mu(y, \tau) = 0$ (whence $\mu(\sigma, \tau) = 0$).

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Suppose τ has at least two occurrences of σ and that $|\tau| - |\sigma| \geq 3$. Assume that $'\tau'$ lies in $[\sigma, \tau]$. Then, if $[\sigma, \tau]$ has no carrier element, we have $\mu(\sigma, \tau) = 0$. Otherwise, $\mu(\sigma, \tau) = \mu(\sigma, \mathcal{C})$.

Proof. We compute $\mu(\sigma, \tau)$ from top to bottom.

$$z \in (C' \setminus {}'C) \cup ({}'C \setminus C') \Rightarrow \mu(z, \tau) = 0.$$

If $\mathcal{C}$ doesn't exist, then $y < {}'\tau' \Rightarrow \mu(y, \tau) = 0$ (whence $\mu(\sigma, \tau) = 0$).

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If $\mathcal{C}$ exists, then

$$\mu(\sigma,\tau) = -\sum_{\sigma<z\leq\tau}\mu(z,\tau) =^{(*)} -\sum_{\sigma<z\leq\mathcal{C}}\mu(z,\tau)$$

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When $z \leq \mathcal{C}$, we get

$$\mu(z,\tau) = -\sum_{z<t\leq\tau}\mu(t,\tau) =^{(*)} -\sum_{z<t\leq\mathcal{C}}\mu(t,\tau)$$

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As a consequence of the previous theorem, we can prove that $\mu(\sigma,\tau) = 0$ happens very often.

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Suppose σ occurs in τ but that the first two (or the last two) letters of τ are not involved in any occurrence of σ. Then $\mu(\sigma,\tau) = 0$.

For the proof, just compute the carrier element $\mathcal{C}$ of $[\sigma,\tau]$, then the carrier element of $[\sigma,\mathcal{C}]$, and so on, until we get $\mathcal{C}' \in [\sigma,\tau]$ such that $[\sigma,\mathcal{C}']$ has no carrier element. Such a $\mathcal{C}'$ is called the *socle* of $[\sigma,\tau]$.

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Some examples

1. $$\begin{array}{l}\sigma = 321 \\ \tau = 431825976\end{array} \rightsquigarrow \mathcal{C} = 321 = \sigma \Rightarrow \mu(\sigma,\tau) = \mu(\sigma,\sigma) = 1$$

2. $$\begin{array}{l}\sigma = 231 \\ \tau = 2\,5\,7\,1\,4\,8\,9\,3\,6\,10\end{array} \rightsquigarrow \mathcal{C} = 245136 \Rightarrow \mu(\sigma,\tau) = \mu(\sigma,\mathcal{C})$$

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Further consequences

Our results allow us to recursively compute $\mu(\sigma, \tau)$.

However, it would be nice to deduce $\mu(\sigma, \tau)$ from a simple inspection of the permutations σ and τ.

In this direction, we only have some partial results, which we list below.

In the following, we denote with x the sum of the lengths of the tails of τ with respect to σ. So $x = 0$ means that τ has two occurrences of σ, one at each end; $x = 1$ means that τ has one occurrence of σ at one end and a tail of length 1 at the other end; finally, $x = 2$ means that τ has two tails of length 1 each.

For simplicity, in what follows we will always assume that, when $x = 1$, τ has an occurrence of σ at its right end (and thus a tail of length 1 at its left end). In case σ appears at the left end of τ, we simply have to replace each occurrence of the word "suffix" with the word "prefix" in all the following propositions.

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Our first result says that it is very difficult for the Möbius function to take the value $(-1)^{x+1}$.

Proposition

1. *Suppose $x = 0$. If τ does not have a non-monotone suffix of length $|\sigma| + 1$, then $\mu(\sigma, \tau) \neq -1$.*
2. *Suppose $x = 1$. If τ does not have a suffix of length $|\sigma| + 1$ whose suffix of length $|\sigma| - 1$ is monotone, then $\mu(\sigma, \tau) \neq 1$.*
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Denote with ω the longest bifix of τ containing σ and having length $\leq |\sigma| + 2$. Moreover, denote with α (resp. β) the shortest prefix (resp. suffix) pattern of τ containing the first (resp. last) two occurrences of ω. If ω exists and $\alpha \neq \beta$, then $\mu(\sigma, \tau) = 0$.

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This comes as no surprise, since factor order on words is very similar to consecutive pattern containment in permutations.

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(Sagan and Willenbring, 2011) Let $A = \{a, b\}$. There is an order isomorphism from the poset A^* endowed with the factor order to the subposet of the consecutive pattern poset consisting of all permutations that avoid the patterns 213 and 231.

So, what about factor order on alphabet of cardinality greater than 2? Is it possible to find an order isomorphism with suitable subposets (hopefully described in terms of avoidance of patterns) of the consecutive pattern poset?

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