Some Adin-Roichman-Mansour type identities

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Outline of this talk

- Identities of Ardin-Roichman & Mansour
- Some ARM identities
  - $\cdot$ $\mathrm{Alt}_n(321)$
  - $\cdot$ $\mathrm{Bax}_n(123)$
  - $\cdot$ $\mathrm{DS}_n(312)$
- Idea of proof
- Problems & Discussions

- Part of these works are joint with Fu, Pan, Yan.

Part 1

Adin-Roichman-Mansour type identities

Pattern avoiding permutations

- π is σ-avoiding:= (we know that)...

  25134 is 321-avoiding.

  · Let $\mathfrak{S}_n(321)$ be the 321-avoiding permutations in $\mathfrak{S}_n$.

Theorem [Simion, Schmidt, 1985]

$$\sum_{\pi \in \mathfrak{S}_n(321)} 1 = c_n,$$

where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number.

Sign-balance

- Amazingly, the signed counting is also a Catalan numbers.

Theorem [Simion, Schmidt, 1985]

$$\sum_{\pi\in\mathfrak{S}_n(321)} (-1)^{\mathrm{inv}(\pi)} = \begin{cases} c_{\frac{n-1}{2}}, \\ 0, \end{cases}$$

i.e.,

Theorem [Simion, Schmidt, 1985]

$$\begin{cases} \displaystyle\sum_{\pi\in\mathfrak{S}_{2n+1}(321)} (-1)^{\mathrm{inv}(\pi)} = \sum_{\pi\in\mathfrak{S}_n(321)} 1. \\ \displaystyle\sum_{\pi\in\mathfrak{S}_{2n}(321)} (-1)^{\mathrm{inv}(\pi)} = 0. \end{cases}$$

Adin-Roichman identities

- In 2004, Adin and Roichman gave a refinement.

$$\begin{aligned} \mathrm{ldes}(\pi) &:= \text{last descent of } \pi \\ &= \max\{1 \le i \le n-1 : \pi(i) > \pi(i+1)\} \\ &\quad (\mathrm{ldes}(\pi) := 0 \text{ if } \pi = \mathrm{id}) \end{aligned}$$

$\cdot\ \mathrm{ldes}(216534) = 4.$

Adin-Roichman's identities

Theorem [Adin, Roichman, 2004, SLC]

$$\begin{cases} \displaystyle\sum_{\pi\in\mathfrak{S}_{2n+1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}(\pi)} = \sum_{\pi\in\mathfrak{S}_n(321)} q^{2\mathrm{ldes}(\pi)} & (n \geq 0). \\ \displaystyle\sum_{\pi\in\mathfrak{S}_{2n}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}\pi} = (1-q) \sum_{\pi\in\mathfrak{S}_n(321)} q^{2\mathrm{ldes}(\pi)} & (n \geq 1). \end{cases}$$

n	1	2	3	4	5
$\sum(-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}(\pi)}$	1	$1-q$	1	$(1-q)(1+q^2)$	$1+q^2$
$\sum q^{\mathrm{ldes}(\pi)}$	1	$1+q$	—	—	—

· When $q=1$, they reduces to Simon-Schmidt's identities.

Mansour's identities

At the same time, Mansour consider the $\mathfrak{S}_n(132)$.

$$\begin{aligned} \mathrm{find}(\pi) &:= \text{the index of the letter '1' in } \pi \\ &= \pi^{-1}(1) \end{aligned}$$

$\cdot$ $\mathrm{find}(216534) = 2.$

Theorem [Mansour, 2004, SLC] For $n \geq 1$,

$$\begin{cases} \displaystyle\sum_{\pi \in \mathfrak{S}_{2n+1}(132)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}(\pi)} = \sum_{\pi \in \mathfrak{S}_{2n+1}(132)} q^{\mathrm{find}(\pi)-1}. \\ \displaystyle\sum_{\pi \in \mathfrak{S}_{2n}(132)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{ldes}\pi} = (1-q) \sum_{\pi \in \mathfrak{S}_{n}(132)} q^{2(\mathrm{find}(\pi)-1)}. \end{cases}$$

$2n$ **reduced to** n **phenomenon**

- The Adin-Roichman-Mansour identities are essentially

  "$2n$ reduces to n phenomena"

  · "Signed enumerator on size $2n$" $=$ "enumerator on size n".

- "ARM-type identities"

  · bending the ARM — folding in half and double in thickness!

- **In this talk**, we present some new instances.

Part 2

Alternating permutations (321)

Alternating permutations

- $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is alternating if
$$\pi_1 > \pi_2 < \pi_3 > \pi_4 < \ldots$$

- $\mathrm{Alt}_n :=$ set of Alternating permutations of length n.
  - $\cdot\ \sum_{n\geq 0} |\mathrm{Alt}_n| \frac{x^n}{n!} = \tan x + \sec x$.
  - $\cdot\ |\mathrm{Alt}_n|_{n\geq 0} = 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots$

- $\mathrm{Alt}_n(321) :=$ Alt_n and avoiding 321.

Theorem [Deutsch, Reifegerste 2003, Mansour 2003]
$$|\mathrm{Alt}_{2n}(321)| = |\mathrm{Alt}_{2n-1}(321)| = \frac{1}{n+1}\binom{2n}{n}$$

$\cdot\ |\mathrm{Alt}_6(321)| = 5$. (214365, 215364, 314265, 315264, 415263).
$\cdot\ |\mathrm{Alt}_5(321)| = 5$. (21435, 21435, 31425, 31524, 41523).

Signed counting

- Motivation: signed counting of $\mathrm{Alt}_{2n}(321)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\sum(-1)^{\mathrm{inv}}$	1	1	-1	0	0	-1	1	0	0	2	-2	0	0	-5	5

- **Theorem** [Pan, Ting, 2011]

· $$\sum_{\mathrm{Alt}_{4n+2}(321)} (-1)^{\mathrm{inv}(\pi)} = (-1)^{n+1} \sum_{\mathrm{Alt}_{2n}(321)} 1$$

· $$\sum_{\mathrm{Alt}_{4n+1}(321)} (-1)^{\mathrm{inv}(\pi)} = (-1)^{n} \sum_{\mathrm{Alt}_{2n}(321)} 1$$

· $$\sum_{\mathrm{Alt}_{4n}(321)} (-1)^{\mathrm{inv}(\pi)} = 0$$

· $$\sum_{\mathrm{Alt}_{4n-1}(321)} (-1)^{\mathrm{inv}(\pi)} = 0$$

ARM on $\mathrm{Alt}_n(321)$

· It is so similar to Simion-Schmidt's result.

  · Hence, it is natural to seek the ARM identities.

• **Theorem** [—, 2012]

· $$\sum_{\mathrm{Alt}_{4n+2}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n+1} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot\mathrm{lead}(\pi)}$$

· $$\sum_{\mathrm{Alt}_{4n+1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot\mathrm{lead}(\pi)}$$

· $$\sum_{\mathrm{Alt}_{4n}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n+1}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{lead}(\pi)-1)}$$

· $$\sum_{\mathrm{Alt}_{4n-1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^{n}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{lead}(\pi)-1)}$$

Another

· 'Ending' ($\mathrm{end}(\pi) := \pi_n$) also works!

• **Theorem** [—, 2012]

· $$\sum_{\mathrm{Alt}_{4n+2}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n+1} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot \mathrm{end}(\pi)}$$

· $$\sum_{\mathrm{Alt}_{4n+1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n} \sum_{\mathrm{Alt}_{2n}(321)} q^{2\cdot \mathrm{end}(\pi)}$$

· $$\sum_{\mathrm{Alt}_{4n}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{end}(\pi)-1)}$$

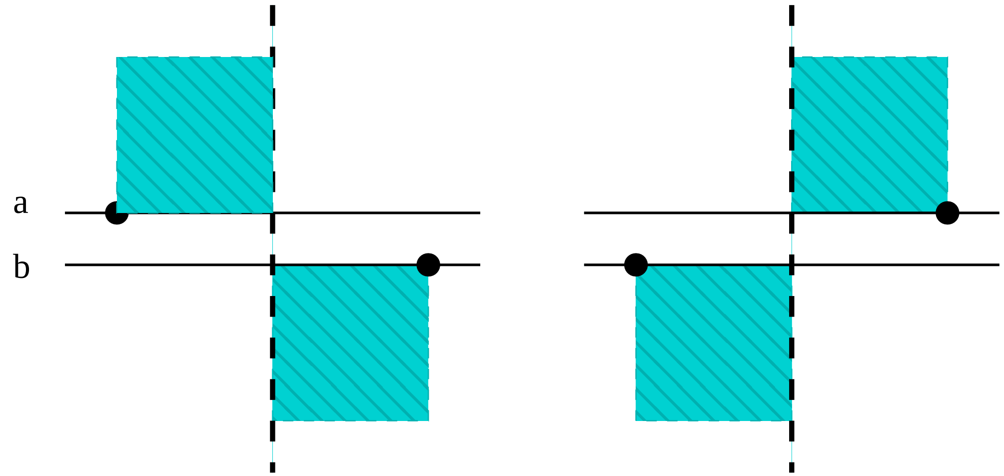
· $$\sum_{\mathrm{Alt}_{4n-1}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{end}(\pi)} = (-1)^{n+1}(1-q) \sum_{\mathrm{Alt}_{2n}(321)} q^{2(\mathrm{end}(\pi)-1)}$$

Part 3

Baxter permutations (321)

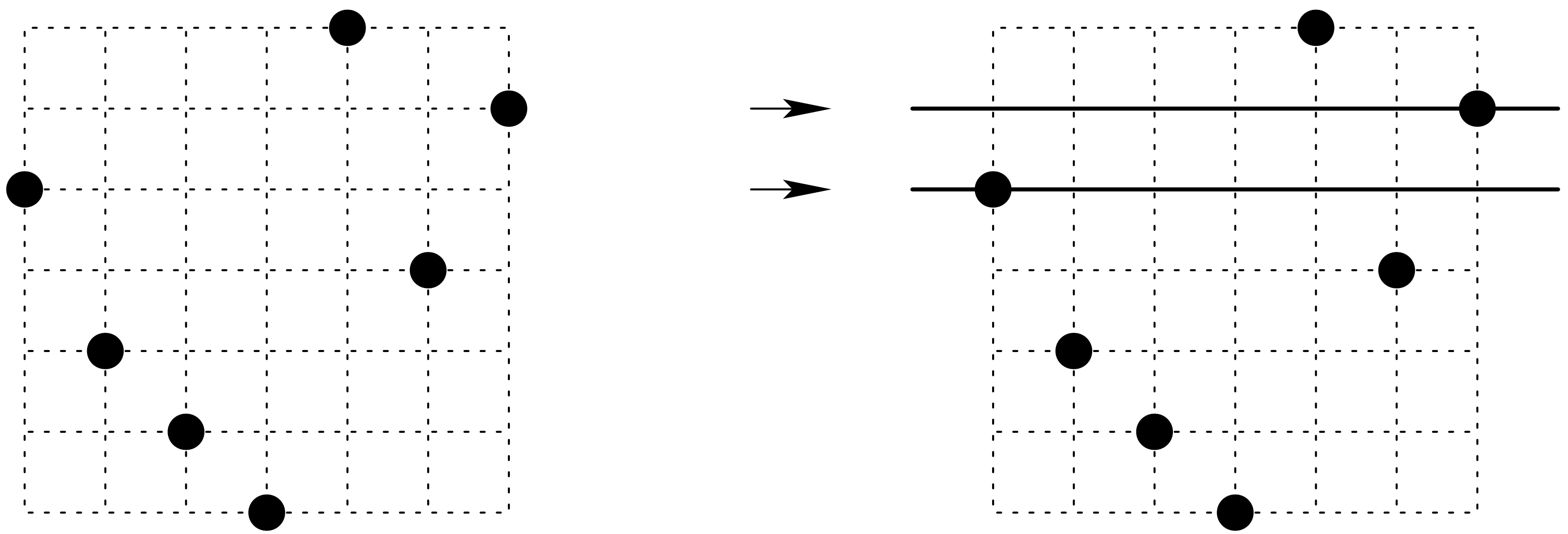
Baxter permutations

- $\pi \in \mathfrak{S}_n$ is Baxter if for all $1 \leq a < b < c < d \leq n$,
  - if $\pi_a + 1 = \pi_d$ and $\pi_b > \pi_d$, then $\pi_c > \pi_d$.
  - if $\pi_d + 1 = \pi_a$ and $\pi_c > \pi_a$, then $\pi_b > \pi_a$.
- For every 2 black dots whose height differ by 1,
  - ...the broken line can be found,
  - ...such that dots in between are in shaded area.



Baxter permutation

- 5321746 is **not** Baxter.



$\cdot$ $\mathrm{Bax}_n :=$ set of Baxter permutations of length n.

Theorem [Chung, Graham, Hoggatt, Kleitman]

$$|\mathrm{Bax}_n| = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}.$$

Baxter permutation avoiding 123

· $\mathrm{Bax}_n(123) :=$ Bax_n and avoid 123.

Theorem [Mansour, Vajnovszki, 2007]

$$\sum_{n\geq 0} |\mathrm{Bax}_n(123)| z^n = \frac{1-2z+z^2}{1-3z+2z^2-z^3}.$$

· $|\mathrm{Bax}_n(123)|_{n\geq 0} = 1, 1, 2, 5, 12, 28, 65, \ldots.$

· $|\mathrm{Bax}_n(123)|_{n\geq 0} = p_{3n}$, a Padovan number, defined by

$$(p_0, p_1, p_2) = (1, 0, 0) \text{ and } p_n = p_{n-2} + p_{n-3}.$$

· $|\mathrm{Bax}_n(321)| = |\mathrm{Bax}_n(123)|$

ARM on $\mathrm{Bax}_n(321)$

- We have found an ARM-type identity on $\mathrm{Bax}_n(321)$.

Theorem [—, 2012] For $n \geq 0$, we have

$$\sum_{\pi \in \mathrm{Bax}_{2n+1}(321)} (-1)^{\mathrm{maj}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{des}(\pi)} = p \cdot \sum_{\pi \in \mathrm{Bax}_n(321)} p^{2 \cdot \mathrm{fix}(\pi)} q^{2\mathrm{des}(\pi)}.$$

· The sign is controlled by maj (somewhat surprising)

On Bax$_{2n}(321)$

- For the even length, the corresponding ARM identity is a sum.

Theorem [—, 2012] For $n \geq 0$, we have

$$\sum_{\pi \in \mathrm{Bax}_{2n}(321)} (-1)^{\mathrm{maj}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{des}(\pi)}$$

$$= (-1)^n \sum_{\pi \in \mathrm{Bax}_n(321)} p^{2 \cdot \mathrm{fix}(\pi)} q^{2\mathrm{des}(\pi)} - q \sum_{i=0}^{n-1} \sum_{\pi \in \mathrm{Bax}_i(321)} p^{2 \cdot \mathrm{fix}(\pi)} q^{2\mathrm{des}(\pi)}.$$

Part 4

Double Simsun permutations (312)

Simsun permutations

- $\pi \in \mathfrak{S}_n$ is **simsun** if for all $1 \leq k \leq n$,
  - $\cdot$ π restricted to $\{1, 2, \ldots k\}$ has no double descent.
  - $\cdot$ 6274351 is **not** simsun,
    - $\cdot$ 2431 has double descent 431.

- Let $|\mathrm{SS}_n|$ be the simsun permutations of length n.
  - $\cdot$ **Theorem** [Simion, Sundaram].

$$|\mathrm{SS}_n| = |\mathrm{Alt}_{n+1}|.$$

Double simsun permutations

- π is **double simsun** if both π and π^{-1} are simsun.
  - 51324 is simsun but **not** double simsun,
    - since $(51324)^{-1} = 24351$ is not simsun.
- Let $|\mathrm{DS}_n|$:= double simsun permutations of length n.
  - $|\mathrm{DS}_n|$ is still unknown. However,....

Theorem [Chuang, Eu, Fu, Pan, Fundamenta Informaticae, to appear].

$$\begin{aligned}
|\mathrm{DS}_n(123)| &= 2, \\
|\mathrm{DS}_n(132)| &= S_n, \\
|\mathrm{DS}_n(213)| &= S_n, \\
|\mathrm{DS}_n(231)| &= 2^{n-1}, \\
|\mathrm{DS}_n(312)| &= 2^{n-1}, \\
|\mathrm{DS}_n(321)| &= \tfrac{1}{n+1}\tbinom{2n}{n},
\end{aligned}$$

- where S_n is the ‘RNA secondary structrue number’.

ARMs on $\mathbf{DS}_n(312)$

Theorem [—, 2012]. For $n \geq 1$, we have

$$\sum_{\pi \in \mathrm{DS}_{2n+2}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{fix}(\pi)} = (-1+q^2) \sum_{\pi \in \mathrm{DS}_n(312)} q^{2\mathrm{fix}(\pi)}.$$

Theorem [—, 2012]. For $n \geq 2$, we have

$$\sum_{\pi \in \mathrm{DS}_{2n-1}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{lead}(\pi)} = \frac{2}{q(1+q^2)} \sum_{\pi \in \mathrm{DS}_n(312)} q^{2\mathrm{lead}(\pi)}.$$

Part 5

Idea of Proof

Part 5-1

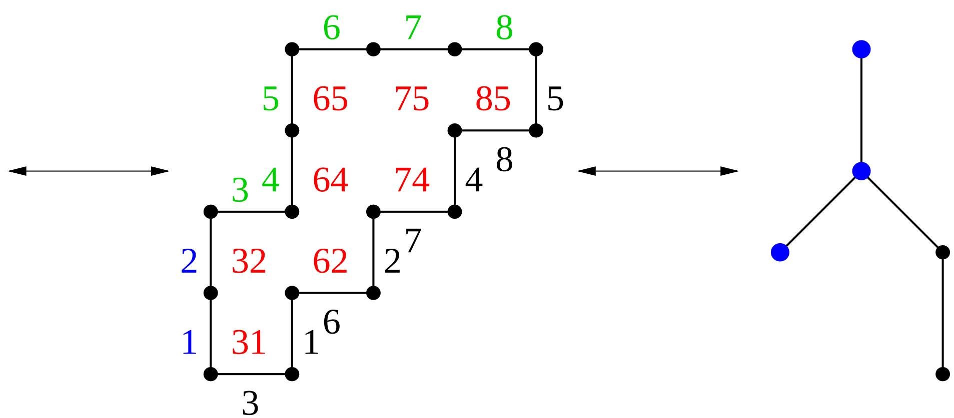
Sketch Proof: $\mathrm{Alt}_n(321)$

$\textbf{Alt}_{4n+2}(321)$ Sketch Proof

- **Step 1**: Map everything to trees. $\mathrm{Alt}_{2n} \longleftrightarrow \mathrm{T}_n$
  - T_n := plane trees with n edges.
    - $\mathrm{hsum}(T)$:= sum of heights of all nodes of T.
    - $\mathrm{lmp}(T)$:=nodes on the leftmost path of T

31627485

31, 32, 62, 64
65, 74, 75, 85



$$\mathrm{inv}(\pi) \longleftrightarrow \mathrm{hsum}(T)$$

$$\mathrm{lead}(\pi) \longleftrightarrow \mathrm{lmp}(T)$$

$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

- **Step 2**: Look at idenities on trees.

We are to prove

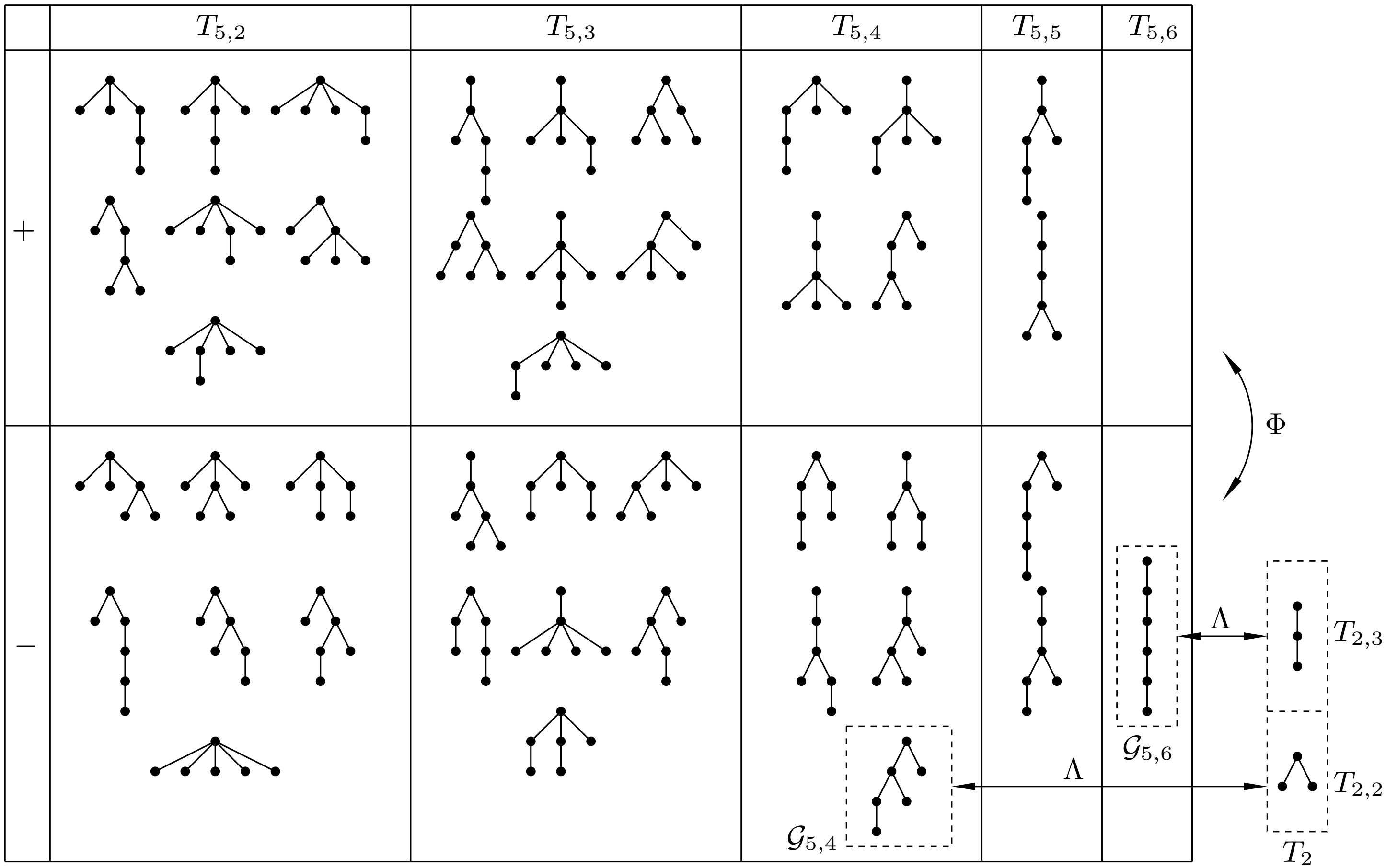
$$\sum_{\mathrm{Alt}_{10}(321)} (-1)^{\mathrm{inv}(\pi)} q^{\mathrm{lead}(\pi)} = (-1)^3 \sum_{\mathrm{Alt}_4(321)} q^{2\cdot\mathrm{lead}(\pi)}$$

which becomes

$$\sum_{\mathrm{T}_5} (-1)^{\mathrm{hsum}(T)} q^{\mathrm{lmp}(T)} = (-1) \sum_{\mathrm{T}_2} q^{2\cdot\mathrm{lmp}(T)}$$

$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

- **Step 3**: Devise an involution Φ (and a bijection Λ).

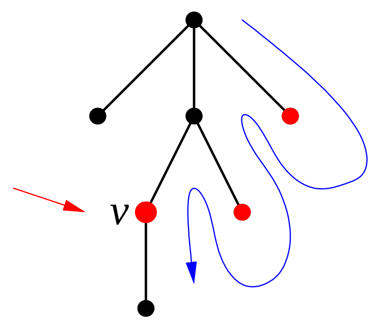


$$\sum_{\mathrm{T}_5} (-1)^{\mathrm{hsum}(T)} q^{\mathrm{lmp}(T)} = (-1) \sum_{\mathrm{T}_2} q^{2\cdot \mathrm{lmp}(T)}$$

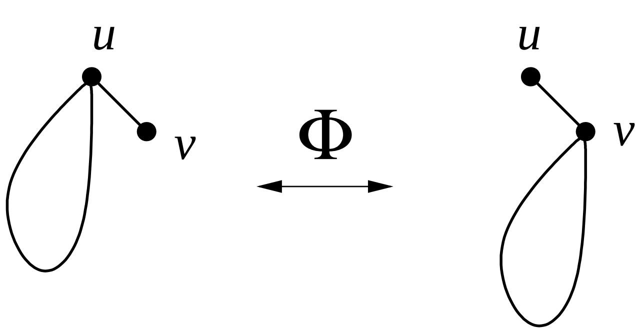
$\mathrm{Alt}_{4n+2}(321)$ Sketch Proof

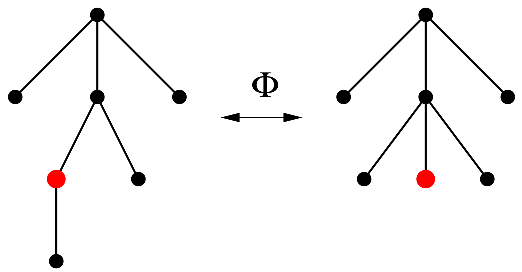
- Φ: an involution (to cancel w.r.t. hsum).
  - Find the last illegal vertex v via postorder
    - illegal vertex: (i) “leaf, but is not the first child” or

      (ii) “inner vertex, but is the first child".



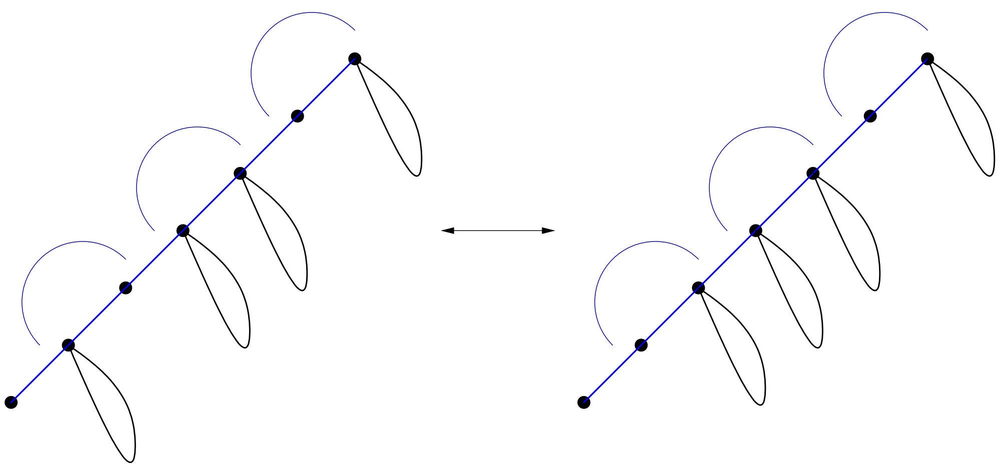
  - Do the following (hsum will change sign, lmp keep unchanged).



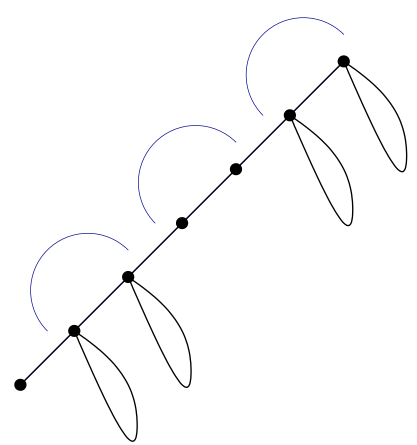


$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

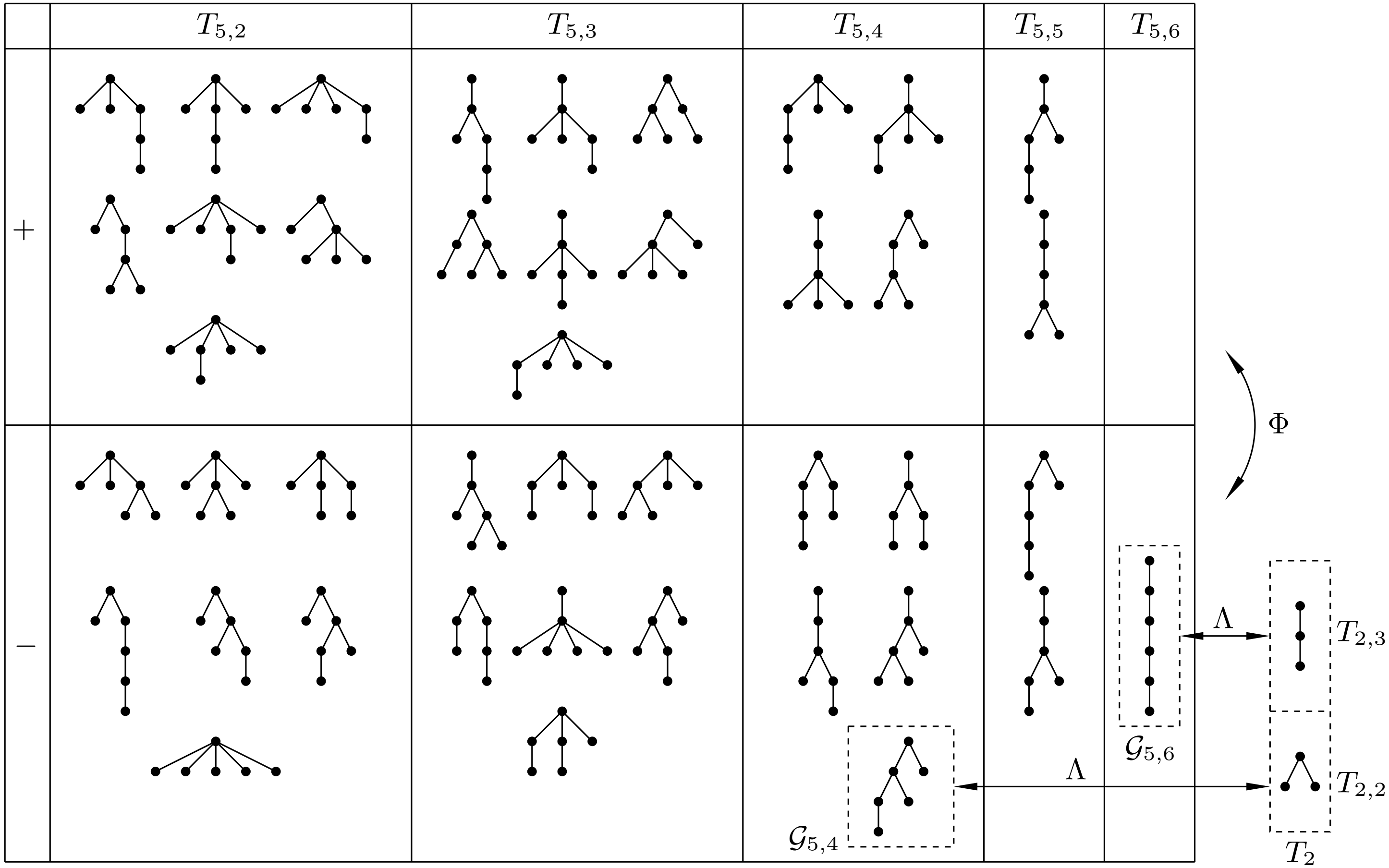
· If $\nexists$ illegal point, change the last (tree, vertex) $\longleftrightarrow$ (vertex, tree).



· If $\nexists$ illegal point, $\nexists$ (tree, vertex), $\nexists$ (vertex, tree), $\Rightarrow$ a fix point.



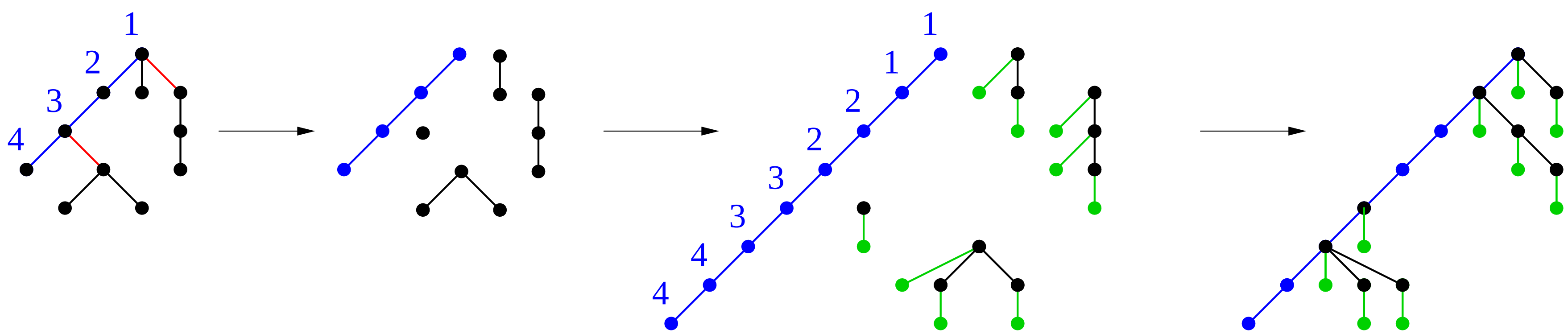
$\textbf{Alt}_{4n+2}(321)$ Sketch Proof



- The remaining task is to map fix points to T_2.
  - We need a bijection Λ: $\mathrm{T}_2 \to \mathrm{T}_5$

$\mathbf{Alt}_{4n+2}(321)$ Sketch Proof

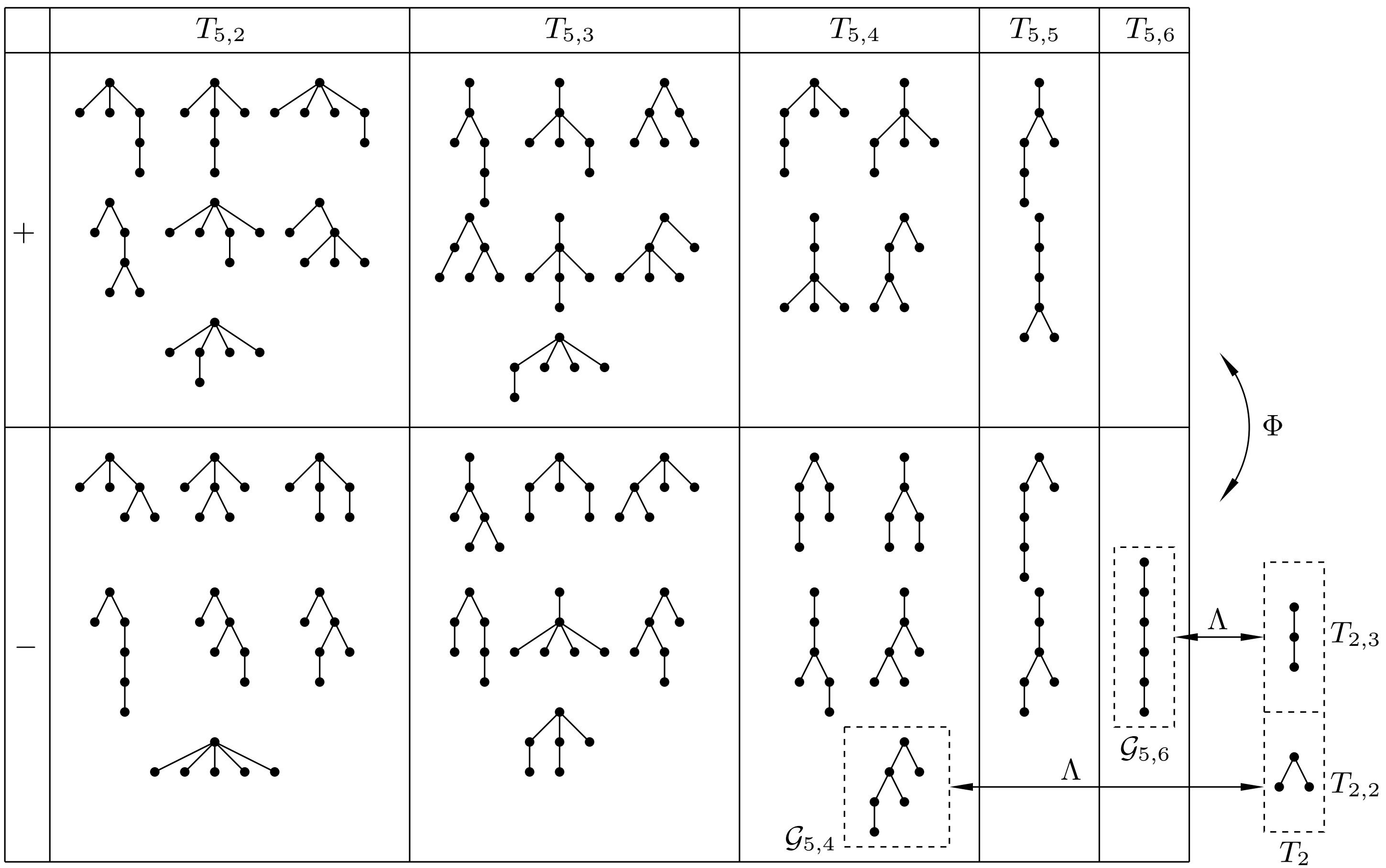
- Λ: a bijection (from T_n to T_{2n+1}).



- Idea:
  - · Double the points on the left most path.
  - · For each subtree, append a single edge on each vertex.

$\textbf{Alt}_{4n+2}(321)$ Sketch Proof

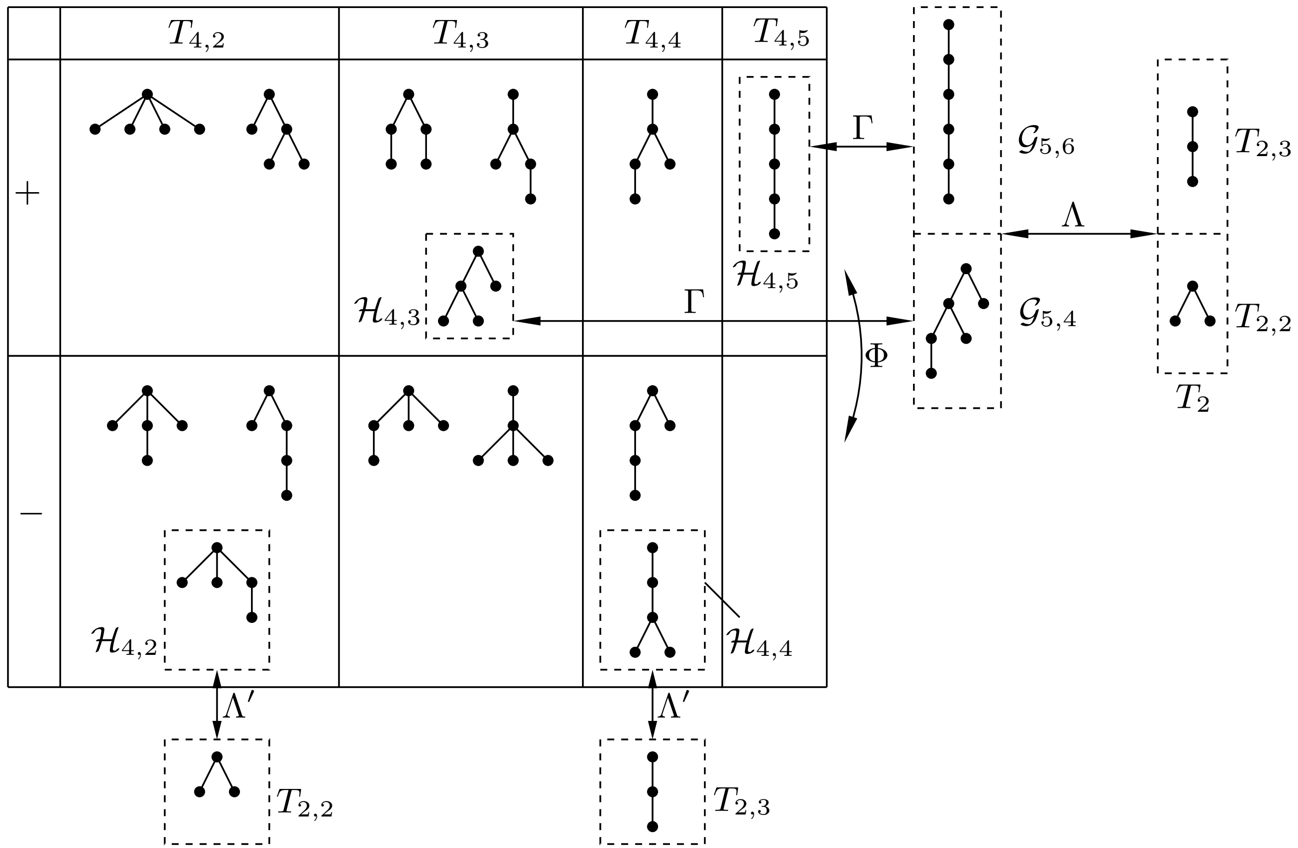
- Let us look the table again:



$$\sum_{\mathrm{T}_5} (-1)^{\color{red}{\mathrm{hsum}}(T)} q^{\color{blue}{\mathrm{lmp}}(T)} = (-1) \sum_{\mathrm{T}_2} q^{2\cdot \color{blue}{\mathrm{lmp}}(T)}$$

$\mathbf{Alt}_{4n}(321)$ **Sketch Proof**

- Proof of $\mathrm{Alt}_{4n}(321)$:



$$\sum_{T_4}(-1)^{\mathrm{hsum}(T)}q^{\mathrm{lmp}(T)} = -\sum_{T_2} q^{2(\mathrm{lmp}(T)-1)} + q\sum_{T_2} q^{2(\mathrm{lmp}(T)-1)}$$

Part 5-2

Very sketched Proof: $\mathrm{Bax}_n(321)$

Sketch proof of Bax$_n(321)$

- Done by multivariable generating functions.
  - Let $b_n := b_n(t,p,q) = \displaystyle\sum_{\pi \in \mathrm{Bax}_n(321)} t^{\mathrm{maj}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{des}(\pi)}$.
  - We are to prove $b_{2n+1}(-1,p,q) = p \cdot b_n(1,p^2,q^2)$.
- Very sketched proof:
  - $g_n := b_n(1,p,q)$
    $$g_n = (2+p)g_{n-1} - (1+2p-q)g_{n-2} + p \cdot g_{n-3}.$$
  - $h_n := b_n(-1,p,q)$
    $$h_{2n+1} = (2+p^2)h_{2n-1} - (1+2p^2-q^2)g_{2n-3} + p^2 \cdot h_{2n-5}.$$
  - $h_{1/3/5}(p,q) = p \cdot g_{0/1/2}(p^2,q^2)$ QED.
- **Problem:** Is there a combinatorial proof?

Part 5-3

Too sketched Proof: $\mathrm{DS}_n(312)$

Sketch proof of DS$_n(312)$

- Done by an involution on compostitions of $[n]$.

- Composition of $[n]$ with DS$_n(312)$: [Eu et al, 2012]
  - $3+4+2 \longleftrightarrow 123|4567|89 \longleftrightarrow 312745698$

- We are to prove

$$\sum_{\pi\in \mathrm{DS}_{2n+2}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{fix}(\pi)} = (-1+q^2) \sum_{\pi\in \mathrm{DS}_n(312)} q^{2\mathrm{fix}(\pi)}.$$

$$\sum_{\pi\in \mathrm{DS}_{2n-1}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathrm{lead}(\pi)} = \frac{2}{q(1+q^2)} \sum_{\pi\in \mathrm{DS}_n(312)} q^{2\mathrm{lead}(\pi)}.$$

- Too sketched proof:
  - Observer what maj, fix, lead means on compositions.
  - Design involutions (...not trivial...).

QED.

Part 6

Discussions

Naive believes were refuted over & over

- ...ARM appears only in $\mathfrak{S}_n(321)$?

  $\rightarrow$ NO, we have $\mathrm{Alt}_n(321)$.

- ...ARM appears only in Catalan family?

  ...sign only controlled by inversion?

  $\rightarrow$ NO, we have $\mathrm{Bax}_n(321)$.

- ...ARM appears only in 321-avoiding?

  $\rightarrow$ NO, we have $\mathrm{DD}_n(312)$.

- ...If ARM appears, then $f(q)$ is a polynomial?

  $\rightarrow$ NO, $f(q) = \frac{2}{q(1+q^2)}$ in $\mathrm{DD}_n(312)$.

General Setting

- Conceptually, we are searching for

$$\mathrm{C}_n,\ (\mathsf{stat}_2, \mathsf{stat}_2)$$

such that essentially the following holds:

$$\sum_{\mathrm{C}_{2n}} (-1)^{\mathsf{stat}_1} q^{\mathsf{stat}_2} = f(q) \sum_{\mathrm{C}_n} q^{2\cdot \mathsf{stat}_2},$$

where $f(q)$ is some rational polynomial.

- In this talk we present
  - $\cdot$ $\mathrm{Alt}_n(321)$, $(\mathrm{inv}, \mathrm{lead})$
  - $\cdot$ $\mathrm{Bax}_n(321)$, $(\mathrm{maj}, \mathrm{des})$
  - $\cdot$ $\mathrm{DS}_n(312)$, $(\mathrm{maj}, \mathrm{fix})$
  - $\cdot$ $\mathrm{DS}_n(312)$, $(\mathrm{maj}, \mathrm{lead})$

Are there others?

- Yes, we have about another 30 nontrivial results.
- E.g.
  - $\mathrm{Alt}_n(321)$, $(\mathrm{chg}, \mathrm{ldes})$
  - $\mathrm{Alt}_n(321)$, $(\mathrm{cochg}, \mathrm{exc})$
  - $\mathrm{Bax}_n(321)$, $(\mathrm{chg}, \mathrm{des})$
  - $\mathrm{Bax}_n(321)$, $(\mathrm{imaj}, \mathrm{fix})$
  - $\mathrm{DS}_n(231)$, $(\mathrm{fix}, \mathrm{end})$
  - $\mathrm{DS}_n(231)$, $(\mathrm{maj}, \mathrm{inv})$
  - ...etc.
- However, ARM identities are relatively rare.
  - C_n, $(\mathrm{stat}_2, \mathrm{stat}_2)$ **seldom** produces ARM.

Discussions

- Encouraging:

  We have a bunch of nontrivial idenities.

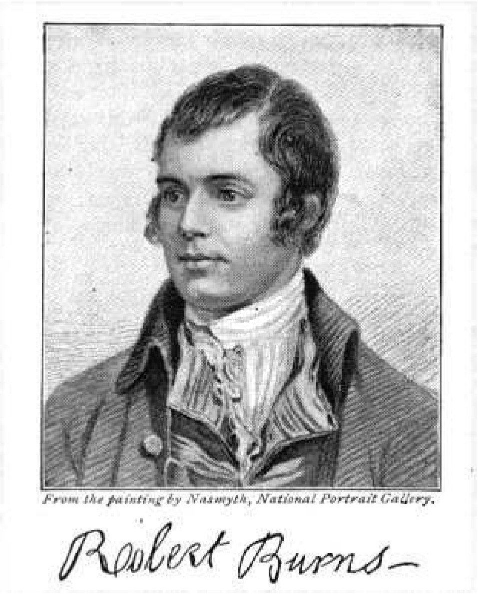
- Disappointing:

  Proofs are done case by case.

- **Question:** Why is there ARM type identities?
  - What is lurking behind?
  - Is there a unifying/systemetic approach?
  - Can it be generalized to, say, Coxter groups?

- We are working on these questions, with some progress.

Thanks



Flow gently, sweet Afton, amang thy green braes,
Flow gently, I'll sing thee a song in thy praise.
— Robert Burns

Welcome any discussions and collaboration!