A combinatorial proof of joint equidistribution of some pairs of permutation statistics

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Combinatorial statistics

Definition

A combinatorial statistic on a set S is a map $\mathbf{f} : S \to \mathbb{N}^m$ for some integer $m \geq 0$. The distribution of $\mathbf{f}$ is the map $\mathrm{d}_{\mathbf{f}} : \mathbb{N}^m \to \mathbb{N}$ with $\mathrm{d}_{\mathbf{f}}(\mathbf{i}) = |\mathbf{f}^{-1}(\mathbf{i})|$ for $\mathbf{i} \in \mathbb{N}^m$, where $|\mathbf{f}^{-1}(\mathbf{i})|$ is the number of objects $s \in S$ such that $\mathbf{f}(s) = \mathbf{i}$.

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We say that statistics $\mathbf{f}$ and $\mathbf{g}$ are equidistributed and write $\mathbf{f} \sim \mathbf{g}$ if $\mathrm{d}_{\mathbf{f}} = \mathrm{d}_{\mathbf{g}}$.

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Classical permutation statistics

Let $S = \mathfrak{S}_n$, $\pi \in \mathfrak{S}_n$.

Descents: Let $\mathtt{Des}(\pi) = \{i : \pi(i) > \pi(i+1)\}$, the descent set of π, and let $\mathtt{des}(\pi) = |\mathtt{Des}(\pi)|$.

$$\ldots ba \ldots, \qquad b > a,$$

$$b = \text{descent top}, \; a = \text{descent bottom}$$

Inversions: Let $\mathtt{Inv}(\pi) = \{(i,j) | \; i < j \text{ and } \pi(i) > \pi(j)\}$ and $\mathtt{inv}(\pi) = |\mathtt{Inv}(\pi)|$.

$$\ldots b \ldots a \ldots, \qquad b > a$$

Eulerian and Mahonian statistics

Definition

Eulerian statistics – same distribution as $\texttt{des}$.

Mahonian statistics – same distribution as $\texttt{inv}$.

Example (Eulerian statistics)

Excedances: $\texttt{exc}(\pi) = |\{i : \pi(i) > i\}|$

Example (Mahonian statistics)

Major index: $$\texttt{maj}(\pi) = \sum_{i \in \texttt{Des}(\pi)} i$$ (MacMahon)

Distribution

Example

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Eulerian and Mahonian partners

- $(\texttt{inv}, \texttt{des}) \sim (\texttt{maj}, \texttt{dmc})$ (Foata; 1977)
- $(\texttt{maj}, \texttt{des}) \sim (\texttt{den}, \texttt{exc})$ (Foata, Zeilberger; 1990)
- $(\texttt{inv}, \texttt{exc}) \sim (\texttt{mad}, \texttt{des})$ (Clarke, Steingrímsson, Zeng; 1997)
- $(\texttt{maj}, \texttt{des}) \sim (\texttt{inv}, \texttt{stc})$ (Scandera; 2002)
- $(\texttt{maj}, \texttt{exc}) \sim (\texttt{aid}, \texttt{des})$ (Shareshian, Wachs; 2007)
- $(\texttt{maj}, \texttt{exc}) \sim (\texttt{inv}, \texttt{lec})$ (Foata, Han; 2008)

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Admissible inversions

Definition

An inversion $(i, j) \in \mathtt{Inv}(\pi)$ is admissible if

- $\pi(j) < \pi(j+1)$, or
- $\pi(j) > \pi(k)$ for some $k \in (i, j)$.

- $\mathtt{Ai}(\pi)$ = set of admissible inversions of π
- $\mathtt{ai}(\pi) = |\mathtt{Ai}(\pi)|$
- $\mathtt{aid}(\pi) = \mathtt{ai}(\pi) + \mathtt{des}(\pi)$

Example

$\mathfrak{S}_3$	123	132	213	231	312	321
ai	0	0	1	0	2	0
des	0	1	1	1	1	2
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aid as a pattern statistic

Admissible inversions can be expressed as a sum of pattern occurrence statistics:

$$\mathtt{ai} = (2\text{-}13) + (3\text{-}12) + (3\text{-}1\text{-}\overline{1'}\text{-}2)$$

So,

$$\mathtt{aid} = (2\text{-}13) + (3\text{-}12) + (3\text{-}1\text{-}\overline{1'}\text{-}2) + (21)$$

Why is aid Mahonian?

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Subexcedant sequences and Lehmer codes

Definition

Let $SE_n = \{(a_1, \ldots, a_n) \ : \ a_i \in [0, n-i]\}$. A sequence $a \in SE_n$ is called a subexcedant sequence of length n.

Obviously, $|SE_n| = n! = |\mathfrak{S}_n|$.

Definition

For a statistic $\mathtt{st}$ on $\mathfrak{S}_n$, let $\mathtt{stcode} : \mathfrak{S}_n \to SE_n$ be such that

$$\mathtt{stcode}(\pi) = (a_1, \ldots, a_n) \implies \mathtt{st}(\pi) = \sum_{i=1}^{n} a_i, \qquad \pi \in \mathfrak{S}_n.$$

We call $\mathtt{stcode}$ a Lehmer code of π with respect to $\mathtt{st}$.

Example

Let $a_i = |\{j > i \ : \ \pi(j) < \pi(i)\}|$. Then we can define $\mathtt{invcode}(\pi) = (a_1, \ldots, a_n)$.

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aidcode

Let $\mathfrak{S}_n^k = \{\pi \in \mathfrak{S}_n \ : \ \pi(1) = k\}$. Define $\psi : \mathfrak{S}_n^k \to \mathfrak{S}_n^{k-1}$ $(k > 1)$:

1. $k\pi_1(k-1)\pi_2 \overset{\psi}{\mapsto} (k-1)\pi_1 k\pi_2$, if $\pi_1 \neq \emptyset$, $\pi_1 \not> k$;
2. $k\pi_1(k-1)\pi_2 \overset{\psi}{\mapsto} (k-1)\pi_1 k\pi_2$, if $\pi_1 \neq \emptyset$, $\pi_1 > k$, $\pi_2 \neq \emptyset$, $\mathtt{F}(\pi_2) > k$;
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Then $\mathtt{aid}(\psi(\pi)) = \mathtt{aid}(\pi) - 1$ and

- $\mathtt{des}(\psi(\pi)) = \mathtt{des}(\pi)$ in Cases 1, 2, 4;
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Thus,

- $\psi^{k-1}(\pi) = 1 \oplus \pi'$ for some $\pi' \in \mathfrak{S}_{n-1}$,
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Let $\mathfrak{S}_n^k = \{\pi \in \mathfrak{S}_n \;:\; \pi(1) = k\}$. Define $\psi : \mathfrak{S}_n^k \to \mathfrak{S}_n^{k-1}$ $(k > 1)$:

1. $k\pi_1(k-1)\pi_2 \overset{\psi}{\mapsto} (k-1)\pi_1 k\pi_2$, if $\pi_1 \neq \emptyset$, $\pi_1 \not> k$;
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Then $\mathtt{aid}(\psi(\pi)) = \mathtt{aid}(\pi) - 1$ and

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Thus,

- $\psi^{k-1}(\pi) = 1 \oplus \pi'$ for some $\pi' \in \mathfrak{S}_{n-1}$,
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Hook factorization

Gessel, 1991: hooks and hook factorization of permutations

Definition

A hook is a descent followed by a (possibly empty) ascent run.

Definition

For $\pi \in S_n$, the hook factorization of π is

$$\pi = \pi_0 \pi_1 \pi_2 \dots \pi_r,$$

where $\pi_1, \dots, \pi_r$ are hooks, $r \geq 0$, and π_0 is a possibly empty ascent run.

Example

$$123|, 1|32, |213, 2|31, |312, 3|21.$$

lec and pix

Definition

$$\mathtt{lec}(\pi) = \sum_{i=1}^{r} \mathtt{inv}(\pi_i), \qquad \mathtt{pix}(\pi) = \text{length of } \pi_0.$$

Example

$\mathfrak{S}_3$	123	132	213	231	312	321
pix	3	1	0	1	0	1
lec	0	1	1	1	2	1
inv	0	1	1	2	2	3

Why pix?

$(\mathtt{maj}, \mathtt{exc}, \mathtt{fix}) \sim (\mathtt{inv}, \mathtt{lec}, \mathtt{pix})$ (Foata, Han; 2008)

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- Rees product,
- poset topology,
- generating functions,
- Lyndon factorization,
- hook factorization,
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Idea: insert letters from right to left, rearranging the resulting permutation suffix after each insertion.

Let S be a set of distinct letters and $k \notin S$ such that $S \cup \{k\}$ is totally ordered. Let σ be a permutation of S. Let $\mathbb{1}$ be the smallest letter in $S \cup \{k\}$. Define a permutation $f(k, \sigma)$ of $S \cup \{k\}$ recursively as follows: $f(\emptyset) = \emptyset$ and

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f(k, A\mathbb{1}B) &= f(k, A)\mathbb{1}B, && \text{if } \sigma = A\mathbb{1}B, k > \mathbb{1}, A \neq \emptyset, B \neq \emptyset,\\
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Example

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Let $\pi = 258697341$.

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- $f(4, 1) = 41$
- $f(3, 41) = 314$
- $f(7, 314) = f(7, 3)14 = 7314$
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Thus, $\phi(258697341) = 258963714$.

Example

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Let $\pi = 258697341$.

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For any permutation π and $\sigma = \phi(\pi)$,

$$\texttt{aidcode}(\sigma) = \texttt{invcode}(\pi),$$
$$\texttt{aid}(\sigma) = \texttt{inv}(\pi),$$
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Example

$\pi = 25|869|73|41$, so $\texttt{invcode}(\pi) = (1,3,5,3,4,3,1,1,0)$,
$\texttt{inv}(\pi) = 21$ and
$\texttt{lec}(\pi) = \texttt{inv}(869) + \texttt{inv}(73) + \texttt{inv}(41) = 1+1+1 = 3$

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Fix partner for $(\mathtt{inv}, \mathtt{lec})$

Q: $(\mathtt{maj}, \mathtt{exc}, \mathtt{fix}) \sim (\mathtt{inv}, \mathtt{lec}, \mathtt{pix}) \sim (\mathtt{aid}, \mathtt{des}, ???)$

A: Define statistic aix on strings of distinct letters as follows:

$$\begin{aligned} \mathtt{aix}\, A\mathbb{1}B &= \mathtt{aix}\, A, && A \neq \emptyset, B \neq \emptyset, \\ \mathtt{aix}\, A\mathbb{1} &= 0, && A \neq \emptyset, \\ \mathtt{aix}\, \mathbb{1}B &= 1 + \mathtt{aix}\, B, \\ \mathtt{aix}\, \emptyset &= 0. \end{aligned}$$

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$\pi = 25|869|73|41$, so $\texttt{pix}(\pi) = \mathrm{length}(25) = 2$

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Remark

The initial increasing run of π has length $\texttt{pix}(\pi)$ or $\texttt{pix}(\pi) + 1$, so

$$0 \leq \texttt{aix}(\pi) \leq \texttt{pix}(\pi) + 1$$

Example

$\mathfrak{S}_3$	123	132	213	231	312	321
aix	3	1	1	0	1	0
des	0	1	1	1	1	2
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`aix` re-defined

It is possible to define `aix` less recursively.

- Given a permutation π find the smallest entry k of π that is an inversion bottom (i.e. occurs somewhere to the right of a larger value). Then $\pi = \pi' k \pi''$ for some strings π', π''.
- If $\pi'' \neq \emptyset$, then delete the suffix $k\pi''$ and iterate the procedure until either $\pi'' = \emptyset$ or no such k exists.
- If no such k exists, then $\texttt{aix}\,\pi$ is the length of the remaining maximal increasing prefix of π.
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aix $\sim$ fix — direct proof

One can see directly that aix and fix have the same distribution.

Let $d_n = |\{\pi \in S_n : \texttt{aix}(\pi) = 0\}|$.

Then, from definition of aix, we can show that

- $|\{\pi \in S_n : \texttt{aix}(\pi) = r\}| = \binom{n}{r} d_{n-r}$
- d_n satisfies the recurrence

$$d_n = \sum_{k=0}^{n-2} \frac{(n-1)!}{k!} d_k, \qquad d_0 = 1, \; d_1 = 0,$$

- so d_n is the number of derangements in $\mathfrak{S}_n$.

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Rawlings major index is a Mahonian statistics that interpolates between maj and inv.

$$\mathtt{Des}_r(\pi) = \{i \in \mathtt{Des}(\pi) \ : \ \pi(i) - \pi(i+1) \geq r\}$$

$$\mathtt{Inv}_r(\pi) = \{(i,j) \in \mathtt{Inv}(\pi) \ : \ \pi(i) - \pi(j) < r\}$$

$$r\mathtt{maj}(\pi) = \sum_{i \in \mathtt{Des}_r(\pi)} i + |\mathtt{Inv}_r(\pi)|$$

Note $1\mathtt{maj} = \mathtt{maj}$, $n\mathtt{maj} = \mathtt{inv}$, $|\mathtt{Inv}_2(\pi)| = \mathtt{des}(\pi^{-1}) = \mathtt{ides}(\pi)$.

- It is known that $(2\mathtt{maj}, \mathtt{ides}) \sim (\mathtt{maj}, \mathtt{exc})$. Find a partner for $\mathtt{fix}$ so that $(2\mathtt{maj}, \mathtt{ides}, ???) \sim (\mathtt{maj}, \mathtt{exc}, \mathtt{fix})$.
- In general, for $2 \leq r \leq n-1$, find the interpolating statistics in $(\mathtt{maj}, \mathtt{exc}, \mathtt{fix}) \sim (r\mathtt{maj}, ???, ???) \sim (\mathtt{inv}, \mathtt{lec}, \mathtt{pix})$.
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Note $1\mathtt{maj} = \mathtt{maj}$, $n\mathtt{maj} = \mathtt{inv}$, $|\mathtt{Inv}_2(\pi)| = \mathtt{des}(\pi^{-1}) = \mathtt{ides}(\pi)$.

- It is known that $(2\mathtt{maj}, \mathtt{ides}) \sim (\mathtt{maj}, \mathtt{exc})$. Find a partner for $\mathtt{fix}$ so that $(2\mathtt{maj}, \mathtt{ides}, ???) \sim (\mathtt{maj}, \mathtt{exc}, \mathtt{fix})$.
- In general, for $2 \leq r \leq n-1$, find the interpolating statistics in $(\mathtt{maj}, \mathtt{exc}, \mathtt{fix}) \sim (r\mathtt{maj}, ???, ???) \sim (\mathtt{inv}, \mathtt{lec}, \mathtt{pix})$.
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