Shape-Wilf-equivalences for vincular patterns

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Goals

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Classify vincular patterns according to Wilf-equivalence.

Vincular patterns

(Also called "generalized patterns" or "dashed patterns".
Definition by examples: Permutation $\pi \in \mathfrak{S}_n$ contains a copy of 23-1 if there are indices $1 \leq i < i+1 < j \leq n$ such that $\pi_i \pi_{i+1} \pi_j \approx 231$.
Example: 24315 contains a copy of 23-1
Example: 31524 avoids 23-1.
Example: 31524 contains a copy of 2-31 (and 2-3-1).

Absence of a dash indicates adjacency required.
Presence of a dash indicates space is allowed.

Vincular patterns in context

“Classical” patterns have all dashes. e.g., 2-3-1.

e.g. Stack-sortable permutations avoid 2-3-1.

“Consecutive” patterns have no dashes. e.g., 231.

e.g. Permutations with no double-descents avoid 321.

Vincular patterns in context

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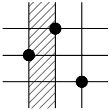
"Consecutive" patterns have no dashes. e.g., 231.

e.g. Permutations with no double-descents avoid 321.

23-1 as a mesh pattern:



Notation

Notation

For a pattern σ, let $\mathfrak{S}_n(\sigma)$ be the σ-avoiding permutations in $\mathfrak{S}_n$ (i.e., those permutations with no copies of σ).
Let $S_n(\sigma) := \left|\mathfrak{S}_n(\sigma)\right|$.

Definition

The patterns α and β are *Wilf-equivalent* if $S_n(\alpha) = S_n(\beta)$ for all $n \geq 0$. Denote this $\alpha \sim \beta$.

Goal

Classify vincular patterns according to Wilf-equivalence.

Previous Work

Classical patterns classified for length $k \leq 7$:
Simion & Schmidt (1985), Babson & West (2001), Stankova & West (2002), Backelin, West, & Xin (2007)

Vincular patterns classified for length $k \leq 3$: Claesson (2001).

Other Wilf-equivalences for vincular patterns:
Kitaev (2005), Elizalde (2006), Kasraoui (2012 preprint)

Current Work

One of the most useful tools for Wilf-classification of classical patterns is "shape-Wilf-equivalence," but this has not be explored for vincular patterns.

Goal

Explore shape-Wilf-equivalence for vincular patterns.

Outline of Talk

- ✓ Introduction
- Shape-Wilf-equivalence
- Equivalence 1
- Equivalences 2 & 3
- Conclusion

Transversals in Young diagrams

A *transversal* π in Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a placement of n rooks in boxes of λ such that there is exactly one rook in every row and column.

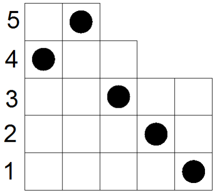


Figure : Transversal $\pi = 45321$ of $\lambda = (5, 5, 4, 3, 3)$.

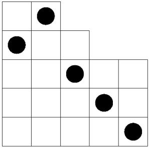
Patterns in transversals

A transversal π of Young diagram λ *contains* σ if

- Underlying permutation π contains σ, *and*
- λ contains the entire box formed by the copy of σ.

Otherwise π avoids σ.

Example: Transversal $\pi = 45321$ of λ



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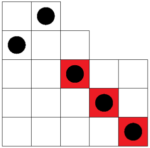
Example: Transversal $\pi = 45321$ of λ contains 321



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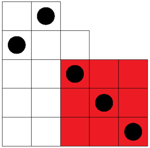

Patterns in transversals

A transversal π of Young diagram λ *contains* σ if

- Underlying permutation π contains σ, *and*
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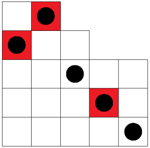
Otherwise π avoids σ.

Example: Transversal $\pi = 45321$ of λ contains 321, but avoids 23-1



Patterns in transversals

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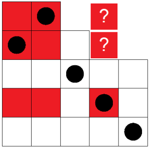


Shape-Wilf-Equivalence

Notation

Let $\mathfrak{S}_\lambda(\sigma)$ be the set of transversals of λ avoiding σ.
Let $S_\lambda(\sigma) := \left|\mathfrak{S}_\lambda(\sigma)\right|$.

Definition

If $S_\lambda(\alpha) = S_\lambda(\beta)$ for all λ, then α and β are *shape-Wilf-equivalent* and we write $\alpha \overset{s}{\sim} \beta$.

Direct sum

The *direct sum* of two permutations, $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_\ell$, is the length-$(k+\ell)$ permutation $\alpha \oplus \beta$, formed by placing β above and to the right of α.

Example: $312 \oplus 2413 = 312\,5746$.

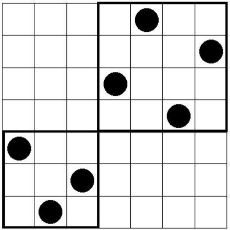


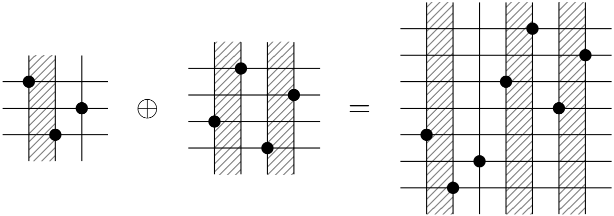
Figure : $312 \oplus 2413 = 312\,5746$

Direct sum for vincular patterns

The *direct sum* of two vincular patterns α and β is the vincular pattern $\alpha \oplus \beta$, formed by placing β above and to the right of α *and inserting a dash between* α *and* β.

Example: $31\text{-}2 \oplus 24\text{-}13 = 31\text{-}2\text{-}57\text{-}46$.

As mesh patterns:



Shape-Wilf-equivalence and direct sums

Backelin, West, and Xin show that shape-Wilf-equivalence combines well with direct sums for classical patterns.

Lemma (Backelin, West, Xin, 2007)

For classical patterns α, β, and σ, if $\alpha \overset{s}{\sim} \beta$ then $\alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$.

Does this hold when α, β, σ are vincular patterns?

Shape-Wilf-equivalence and direct sums

Backelin, West, and Xin show that shape-Wilf-equivalence combines well with direct sums for classical patterns.

Lemma (Backelin, West, Xin, 2007)

For classical patterns α, β, *and* σ, *if* $\alpha \overset{s}{\sim} \beta$ *then* $\alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$.

Does this hold when α, β, σ are vincular patterns? Yes.

Lemma (B. 2012)

For vincular patterns α, β, *and* σ, *if* $\alpha \overset{s}{\sim} \beta$ *then* $\alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$.

(Also true for certain mesh patterns α, β, and σ.)

Illustration: $\alpha \overset{s}{\sim} \beta \implies \alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$

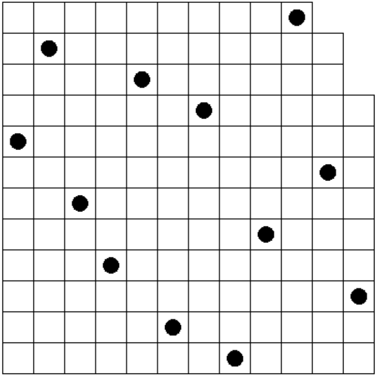


Figure : Start with transversal $\pi \in \mathfrak{S}_\lambda(\alpha \oplus \sigma)$. (Here $\alpha = 1\text{-}2$ and $\sigma = 1\text{-}2$)

Illustration: $\alpha \overset{s}{\sim} \beta \implies \alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$

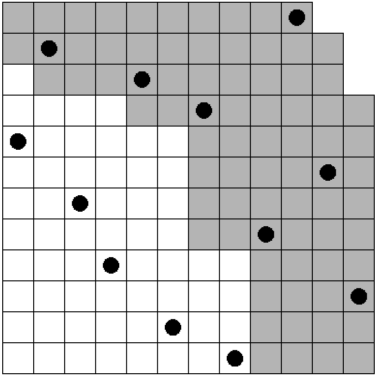


Figure : Color cells white if there is a σ northeast of it. Gray otherwise. (Here, $\sigma = $ 1-2.)

Illustration: $\alpha \overset{s}{\sim} \beta \implies \alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$

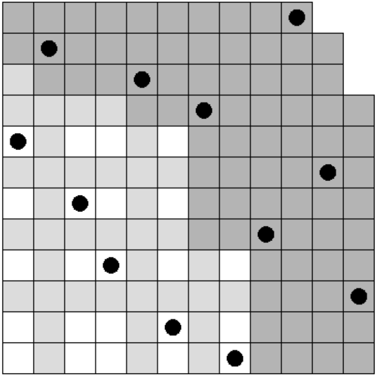


Figure : Color white cells gray if they are in the same row/column as a rook in a gray cell.

Illustration: $\alpha \overset{s}{\sim} \beta \implies \alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$

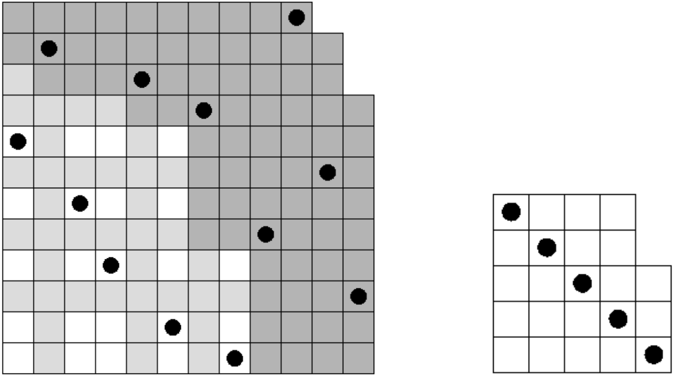


Figure : Remaining rooks in white cells form a α-avoiding transversal of another Young diagram. Use bijection $f : \mathfrak{S}_{\lambda'}(\alpha) \to \mathfrak{S}_{\lambda'}(\beta)$ on white board.

Pairs of shape-Wilf-equivalent classical patterns

Previous work uncovered one family and one sporadic shape-Wilf-equivalence among classical patterns.

Theorem (Backelin, West, Xin, 2007)

$(1\text{-}2\text{-}\cdots\text{-}t) \overset{s}{\sim} (t\text{-}\cdots\text{-}2\text{-}1)$ *for any* $t \geq 1$.

Theorem (Stankova, West, 2002)

$2\text{-}3\text{-}1 \overset{s}{\sim} 3\text{-}1\text{-}2$.

Potential pairs of shape-Wilf-equivalent vincular patterns

A computer search of length 3 vincular shows at most three more potential shape-Wilf-equivalent pairs:

1. $12\text{-}3 \overset{s}{\sim} 21\text{-}3$
2. $1\text{-}23 \overset{s}{\sim} 3\text{-}12$
3. $1\text{-}32 \overset{s}{\sim} 3\text{-}21$

All three are true.

Outline of Talk

- ✓ Introduction
- ✓ Shape-Wilf-equivalence
- Equivalence 1: $12\text{-}3 \overset{s}{\sim} 21\text{-}3$
- Equivalences 2 & 3: $1\text{-}23 \overset{s}{\sim} 3\text{-}12$ and $1\text{-}32 \overset{s}{\sim} 3\text{-}21$
- Conclusion

Equivalence 1: 12-3 $\stackrel{s}{\sim}$ 21-3

Previously: Elizalde (2006), Kitaev (2005): For consecutive patterns α and β, $\alpha \sim \beta \implies \alpha \oplus 1 \sim \beta \oplus 1$.

Theorem (B. 2012)

Let α, β be consecutive patterns. If $\alpha \sim \beta$, then $\alpha \oplus 1 \stackrel{s}{\sim} \beta \oplus 1$.

Corollary

12-3 $\stackrel{s}{\sim}$ 21-3

Illustration of bijection $\mathfrak{S}_\lambda(21 \oplus 1) \to \mathfrak{S}_\lambda(12 \oplus 1)$.

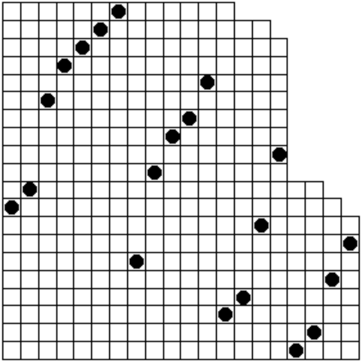


Figure : Start with a transversal $\pi \in \mathfrak{S}_\lambda(\alpha \oplus 1)$.

Illustration of bijection $\mathfrak{S}_\lambda(21 \oplus 1) \to \mathfrak{S}_\lambda(12 \oplus 1)$.

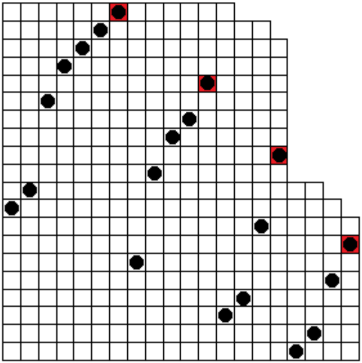


Figure : Identify the right-to-left maxima of π.

Illustration of bijection $\mathfrak{S}_\lambda(21 \oplus 1) \to \mathfrak{S}_\lambda(12 \oplus 1)$.

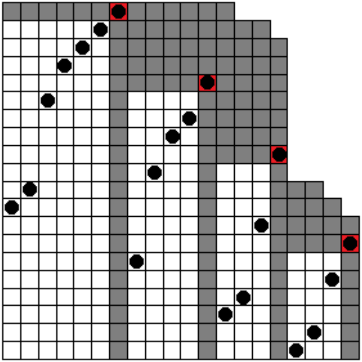


Figure : Dissect π according to right-to-left maxima. Each subword in white cells avoids α.

Illustration of bijection $\mathfrak{S}_\lambda(21 \oplus 1) \to \mathfrak{S}_\lambda(12 \oplus 1)$.

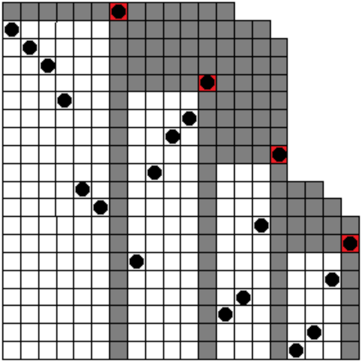
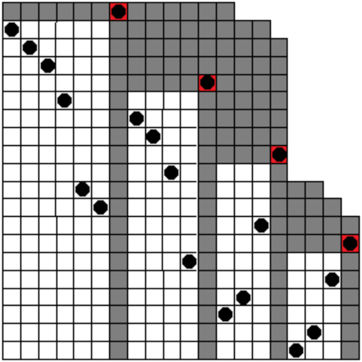


Figure : Use the bijection $\mathfrak{S}_m(\alpha) \to \mathfrak{S}_m(\beta)$ on each subword between right-to-left maxima.

Illustration of bijection $\mathfrak{S}_\lambda(21 \oplus 1) \to \mathfrak{S}_\lambda(12 \oplus 1)$.



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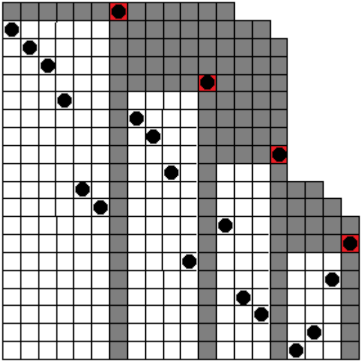

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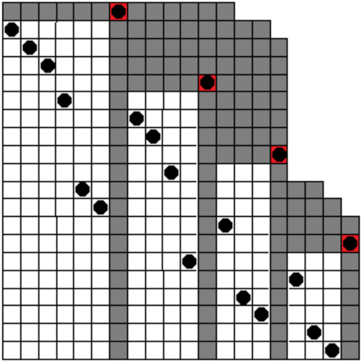


Figure : Use the bijection $\mathfrak{S}_m(\alpha) \to \mathfrak{S}_m(\beta)$ on each subword between right-to-left maxima.

Illustration of bijection $\mathfrak{S}_\lambda(21 \oplus 1) \to \mathfrak{S}_\lambda(12 \oplus 1)$.

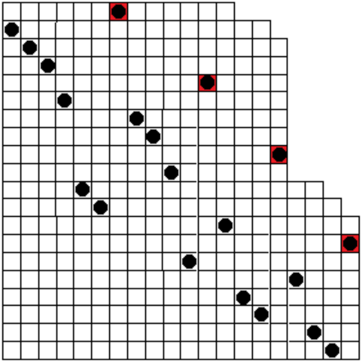


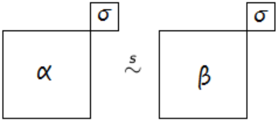
Figure : The end result is a transversal in $\mathfrak{S}_\lambda(\beta \oplus 1)$ (with the same right-to-left maxima).

A significant generalization

We may extend this approach significantly to get:

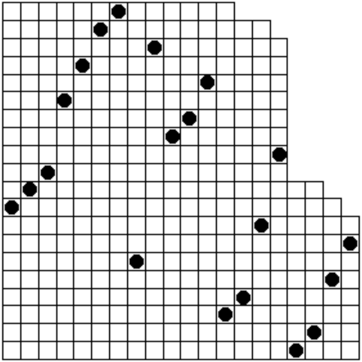
Theorem (B. 2012)

Let α, β, and σ be consecutive patterns. If $\alpha \sim \beta$, then $\alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$.



Changes to the proof: "Right-to-left maxima" get replaced by "right-to-left maximal copies of σ".

Illustration of bijection: $\mathfrak{S}_\lambda(21 \oplus 12) \to \mathfrak{S}_\lambda(12 \oplus 12)$



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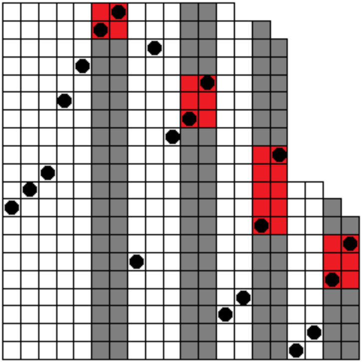

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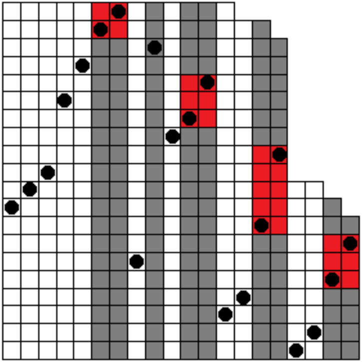

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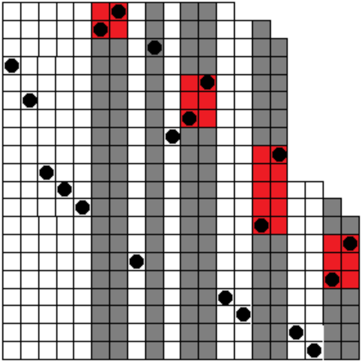

A different generalization

Theorem (B. 2012)

Let α and β be vincular patterns of length k so that both end with k. If $\alpha \sim \beta$ then $\alpha \overset{s}{\sim} \beta$.

Corollary

$12\text{-}3 \overset{s}{\sim} 21\text{-}3$

Corollary

$3124 \overset{s}{\sim} 3214$. *(Elizalde & Noy (2003) proved $3124 \sim 3214$)*

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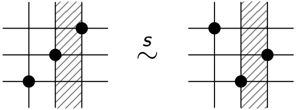
- ✓ Equivalence 1: $12\text{-}3 \stackrel{s}{\sim} 21\text{-}3$
- Equivalences 2 & 3: $1\text{-}23 \stackrel{s}{\sim} 3\text{-}12$ and $1\text{-}32 \stackrel{s}{\sim} 3\text{-}21$
- Conclusion

The other equivalences

We now turn our attention to proving the equivalence 1-23 $\overset{s}{\sim}$ 3-12.



The equivalence 1-32 $\overset{s}{\sim}$ 3-21 is proven similarly.

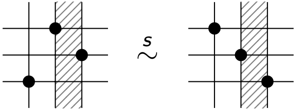
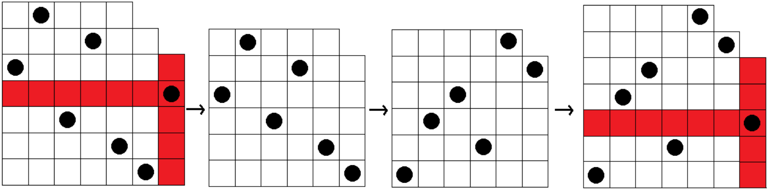


Illustration of bijection: $\mathfrak{S}_\lambda(1\text{-}23) \to \mathfrak{S}_\lambda(3\text{-}12)$

Main Technique: Delete the last letter and use induction.



End with a stronger result:

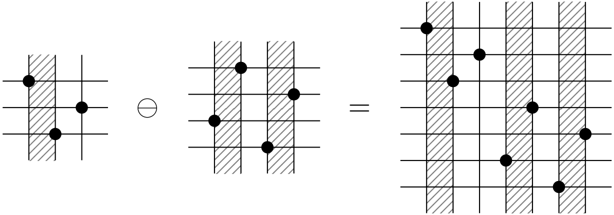
$$S_\lambda(1\text{-}23)[a] = \begin{cases} S_\lambda(3\text{-}12)[\lambda_n] & a = 1 \\ S_\lambda(3\text{-}12)[a-1] & 2 \le a \le \lambda_n, \end{cases}$$

where $S[a]$ is the number of $\pi \in S$ ending with a.

Skew-sums

For patterns α, β, form the *skew sum* $\alpha \ominus \beta$ by placing β below and to the right of α and inserting a dash between α and β.

Example 31-2 $\ominus$ 24-13 = 75-6 - 24-13



Generalization of equivalences 2 & 3

Equivalence 2: $1 \oplus 12 \overset{s}{\sim} 1 \ominus 12$
Equivalence 3: $1 \oplus 21 \overset{s}{\sim} 1 \ominus 21$.

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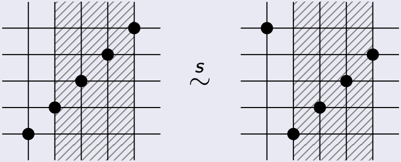
Equivalence 2: $1 \oplus 12 \overset{s}{\sim} 1 \ominus 12$

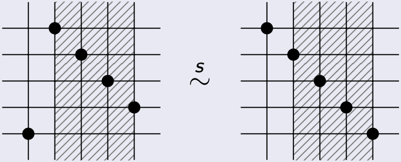
Equivalence 3: $1 \oplus 21 \overset{s}{\sim} 1 \ominus 21$.

Theorem (B. 2012)

$1 \oplus (12 \cdots t) \overset{s}{\sim} 1 \ominus (12 \cdots t)$ *for any* $t \geq 2$



$1 \oplus (t \cdots 21) \overset{s}{\sim} 1 \ominus (t \cdots 21)$ *for any* $t \geq 2$



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- Conclusion

A new Wilf-equivalence class

Corollary

$12\text{-}3\text{-}4 \sim 21\text{-}3\text{-}4 \sim 12\text{-}4\text{-}3 \sim 21\text{-}4\text{-}3$

Proof.

1. $12\text{-}3\text{-}4 = (12 \oplus 1) \oplus 1 \overset{s}{\sim} (21 \oplus 1) \oplus 1 = 21\text{-}3\text{-}4.$
2. $(12\text{-}3\text{-}4)^{rc} = 1\text{-}2\text{-}34 = (1\text{-}2) \oplus 12$
   $(12\text{-}4\text{-}3)^{rc} = 2\text{-}1\text{-}34 = (2\text{-}1) \oplus 12$
   $1\text{-}2 \overset{s}{\sim} 2\text{-}1$, so $(1\text{-}2) \oplus 12 \overset{s}{\sim} (2\text{-}1) \oplus 12.$
3. $(21\text{-}3\text{-}4)^{rc} = (1\text{-}2) \oplus 21 \overset{s}{\sim} (2\text{-}1) \oplus 21 = (21\text{-}4\text{-}3)^{rc}$

□

Wilf-equivalence results

Elizalde / Kitaev proved 1-23-4 $\sim$ 1-32-4. The equivalences 1-23 $\overset{s}{\sim}$ 3-12 and 1-32 $\overset{s}{\sim}$ 3-21 add to this class:

Corollary

3-12-4 $\sim$ 1-23-4 $\sim$ 1-32-4 $\sim$ 3-21-4

There seems to be one more member of this class:

Conjecture

23-1-4 $\sim$ 3-12-4

Wilf-equivalence results

Corollary

1. 123-4 $\sim$ 321-4
2. 213-4 $\sim$ 231-4 $\sim$ 132-4 $\sim$ 312-4
3. 12-3-4 $\sim$ 12-4-3 $\sim$ 21-3-4 $\sim$ 21-4-3
4. 12-34 $\sim$ 12-43 $\sim$ 21-34 $\sim$ 21-43
5. 3-12-4 $\sim$ 1-23-4 $\sim$ 1-32-4 $\sim$ 3-21-4

   Conj: 23-1-4 $\sim$ 3-12-4

More Wilf-equivalence results

Corollary

1. $12\text{-}345 \sim 21\text{-}345 \sim 12\text{-}543 \sim 21\text{-}543$
2. $12\text{-}435 \sim 12\text{-}453 \sim 12\text{-}534 \sim 12\text{-}354 \sim 21\text{-}435 \sim 21\text{-}453 \sim 21\text{-}534 \sim 21\text{-}354$
3. $123\text{-}4\text{-}5 \sim 321\text{-}4\text{-}5$
4. $213\text{-}4\text{-}5 \sim 231\text{-}4\text{-}5 \sim 132\text{-}4\text{-}5 \sim 312\text{-}4\text{-}5$
5. $12\text{-}3\text{-}4\text{-}5 \sim 12\text{-}5\text{-}4\text{-}3 \sim 12\text{-}3\text{-}5\text{-}4 \sim 21\text{-}3\text{-}4\text{-}5 \sim 21\text{-}5\text{-}4\text{-}3 \sim 21\text{-}3\text{-}5\text{-}4$
6. $12\text{-}4\text{-}3\text{-}5 \sim 21\text{-}4\text{-}3\text{-}5$ **Conj:** $12\text{-}3\text{-}4\text{-}5 \sim 12\text{-}4\text{-}3\text{-}5$
7. $12\text{-}5\text{-}3\text{-}4 \sim 12\text{-}4\text{-}5\text{-}3 \sim 21\text{-}5\text{-}3\text{-}4 \sim 21\text{-}4\text{-}5\text{-}3$

Shape-Wilf-equivalent pairs of consecutive patterns

A computer search suggests:

Conjecture

1. $4123 \overset{s}{\sim} 4213$
2. $1432 \overset{s}{\sim} 1342$
3. $2341 \overset{s}{\sim} 2431$

Note: $3124 \overset{s}{\sim} 3214$ discussed previously.

Conclusion

1. The notion of shape-Wilf-equivalence extends nicely to vincular patterns (and mesh patterns in general).
2. If α, β, and σ are vincular patterns, then $\alpha \overset{s}{\sim} \beta \implies \alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$.
3. If α, β, and σ are consecutive patterns, then $\alpha \sim \beta \implies \alpha \oplus \sigma \overset{s}{\sim} \beta \oplus \sigma$.
4. $1 \oplus (12 \cdots t) \overset{s}{\sim} 1 \ominus (12 \cdots t)$ and $1 \oplus (t \cdots 21) \overset{s}{\sim} 1 \ominus (t \cdots 21)$ for any $t \geq 2$.
5. These shape-Wilf-equivalences have many consequences for the Wilf-classification of vincular patterns.

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Thank you.