Up-down ascent sequences and the q-Genocchi numbers

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The Genocchi number G_{2n} for $n \geq 1$ can be defined through its relation with the Bernoulli numbers $G_{2n} = 2(2^{2n} - 1)B_n$ or through its exponential generating function

$$\frac{2t}{e^t + 1} = t + \sum_{n \geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.$$

The median Genocchi numbers H_{2n+1} are defined by $H_{2n+1} = \sum_{i \geq 0} G_{2n-2i} \binom{n}{2i+1}$. The Genocchi numbers have been given combinatorial interpretations by Dumont [?], Dumont and Viennot [?], Burstein et. al. [?], and others.

Zeng and Zhou [?] defined a q-analogue of the Genocchi numbers by defining a q-analogue of the so-called Seidel triangle for the Genocchi numbers by defining polynomials $(g_{i,j}(q))_{i,j \geq 1}$ by $g_{1,1}(q) = g_{2,1}(q) = 1$ and

$$g_{2i+1,j}(q) = g_{2i+1,j-1}(q) + q^{j-1} g_{2i,j}(q), \text{ for } j = 1, 2, \ldots, i+1, \tag{1}$$
$$g_{2i,j}(q) = g_{2i,j+1}(q) + q^{j-1} g_{2i-1,j}(q), \text{ for } j = i, i-1, \ldots, 1, \tag{2}$$

where $g_{i,j}(q) = 0$ if $j < 0$ or $j > \lceil i/2 \rceil$ by convention. They defined the q-Genocchi number $G_{2n}(q)$ and the median q-Genocchi number $H_{2n-1}(q)$ by

$$G_{2n}(q) = g_{2n-1,n}(q) \text{ and } H_{2n-1}(q) = q^{n-2} g_{2n-1,1}(q).$$

We give a new combinatorial interpretation of the elements of the q-analogue of the Seidel triangle in terms of q-counting a up-down ascent sequences. Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [?] to study the problem of enumerating $(\mathbf{2}+\mathbf{2})$-free posets. A sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ is an *ascent sequence of length* n if and only if it satisfies $a_1 = 0$ and $a_i \in [0, 1 + \text{asc}(a_1, \ldots, a_{i-1})]$ for all $2 \leq i \leq n$. Here, for any integer sequence $(a_1, \ldots, a_i)$, the number of *ascents* of this sequence is

$$\text{asc}(a_1, \ldots, a_i) = |\{j : a_j < a_{j+1}\}|.$$

For any $n \geq 1$, we let Asc_n denote the set of all ascent sequences of length n. Then we say that $a = a_1 \ldots a_n \in Asc_n$ is an *up-down* ascent sequence if $a_1 < a_2 > a_3 < a_4 > \cdots$. Let $UDA_n^{(i)}$ denote the set of elements $a = a_1 \ldots a_n \in UDA_n$ such that $a_n = i$. If $n \geq 1$ and $a = a_1 \ldots a_n \in UDA_n$, then we define the weight of a, $w(a)$, by

$$w(a) = \sum_{i=1}^{n-1} (a_i - \chi(i \text{ even})) \tag{3}$$

where for any statement A, $\chi(A) = 1$ is A is true and $\chi(A) = 0$ is A is false. We prove the following theorem.

Theorem 1. *For all* $1 \leq j \leq \lceil i/2 \rceil$,

$$g_{2i,j}(q) = \sum_{a = a_1 \ldots a_{2i+1} \in UDA_{2i+1}^{(j-1)}} q^{w(a)} \text{ and}$$
$$g_{2i+1,j}(q) = \sum_{a = a_1 \ldots a_{2i+2} \in UDA_{2i+2}^{(j)}} q^{w(a)}.$$

It follows for all $n \geq 1$,

$$G_{2n}(q) = g_{2n-1,n}(q) = \sum_{\substack{a = a_1 \ldots a_{2n} \in UDA_{2n} \\ a_{2n} = n}} q^{w(a)}. \tag{4}$$

and

$$H_{2n-1}(q) = q^{n-2} g_{2n-1,1}(q) = q^{n-2} \sum_{\substack{a = a_1 \ldots a_{2n} \in UDA_{2n} \\ a_{2n} = 1}} q^{w(a)}. \tag{5}$$

[1] M. Bousquet-Mélou, A. Claesson, M. Dukes, S. Kitaev: Unlabeled (2+2)-free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A*, **117** Issue 7 (2010), 884–909.

[2] A. Burstein, M. Josuat-Vergés, and W. Stromquist, New Dumont permutations, Pure Math. and Applications, **21** No. 2 (2010), 177–206.

[3] D. Dumont, Interprétations combinatoires des nombres de Genocchi, Duke Math. J., 41 (1974) 305–318.

[4] D. Dumont and X. G. Viennot, A combinatorial interpretation of the Seidel generation of Genocchi numbers, Discrete Math., 6 (1980) 77-87.

[5] X. G. Viennot, Interprétations combinatoires des nombres d'Euler et Genocchi, Séminaire de Théorie des Nombres, Année 1980-1981, exposé no. 11.

[6] J. Zeng and J. Zhou, A q-analog of the Seidel generation of Genocchi numbers, European J. Combin., 27 (2006) 364–381.