Generalized Interval Embeddings

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Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $\mathbb{N}^*$ denote the set of all words over $\mathbb{N}$. Let ϵ denote the empty word. Given words u and v in $\mathbb{N}^*$, we say that u is a factor of v if there are words w_1 and w_2 such that $v = w_1 u w_2$. In such a situation, we say u is a suffix of v if $w_2 = \epsilon$. Given $u = u_1 u_2 \ldots u_\ell \in \mathbb{N}^*$, we define the *norm of* u to be $\Sigma u = u_1 + u_2 + \cdots + u_\ell$ and we define the *length of* u to be $|u| = \ell$. We then allow x and t to be commuting variables and we define the *weight of* u to be $wt(u) = x^{\Sigma u} t^{|u|}$.

Given any poset $\mathcal{P} = (\mathbb{N}, \leq_p)$ and $m, n \in \mathbb{N}$, we let $I^{\mathcal{P}}_{m,\infty} = \{k \in \mathbb{N} : m \leq_p k\}$ and $I^{\mathcal{P}}_{m,n} = \{n \in \mathbb{N} : m \leq_p k \leq_p n\}$. Given any words $u = u_1 \ldots u_k$ and $w = w_1 w_2 \ldots w_\ell$ in $\mathbb{N}^*$, we say that u *embeds into* w *relative to* $\mathcal{P}$, written $u \leq_{\mathcal{P}} w$ if there is a factor $w' = w'_1 w'_2 \ldots w'_k$ of w such that $w'_i \in I^{\mathcal{P}}_{u_i,\infty}$ for every $1 \leq i \leq k$. We define $S^{\mathcal{P}}(u)$ to be the set of all words w that embed u such that the only embedding of u into w occurs at the right end of w, and we set

$$\mathcal{S}^{\mathcal{P}}(u, x, t) = \sum_{w \in S^{\mathcal{P}}(u)} wt(w).$$

Given $u, v \in \mathbb{N}^*$, u and v are $\mathcal{P}$*-Wilf equivalent*, written as $u \sim_{\mathcal{P}} v$, if $\mathcal{S}^{\mathcal{P}}(u, x, t) = \mathcal{S}^{\mathcal{P}}(v, x, t)$. Kiteav, Liese, Remmel and Sagan [1] studied various properties of $\mathcal{P}$-Wilf Equivalence where $\mathcal{P}$ is the standard order on $\mathbb{N}$ and Langley, Liese, and Remmel [2] studied various properties of $\mathcal{P}_k$-Wilf equivalence where $\mathcal{P}_k = (\mathbb{N}, \leq_k)$ and $i \leq_k j$ if and only if $i \equiv j \mod k$ and $i < j$.

We study a generalization $\mathcal{P}$-Wilf equivalence based on intervals. That is, suppose that we are given a poset $\mathcal{P} = (\mathbb{N}, \leq_p)$ and a sequence $\vec{U}$ of intervals $(\{I^{\mathcal{P}}_{m_1,n_1}, I^{\mathcal{P}}_{m_2,n_2}, \ldots, I^{\mathcal{P}}_{m_k,n_k}\}$ where either $m_i \leq_p n_i$ and $m_i, n_i \in \mathbb{N}$ or $m_i \in \mathbb{N}$ and $n_i = \infty$. Then we say that w has an *interval-embedding of* $\vec{U}$ *into* w *relative to* $\mathcal{P}$, denoted $\vec{U} \leq_{\mathcal{P}} w$ if there is a factor $w' = w'_1 w'_2 \ldots w'_k$ of w such that $w'_i \in I^{\mathcal{P}}_{m_i,n_i}$ for every $1 \leq i \leq \ell$. We then define $S^{\mathcal{P}}(\vec{U})$ to be the set of all words $w = w_1 \ldots w_n \in \mathbb{N}^*$ such that $n \geq k$, there is an interval embedding of $\vec{U}$ into the suffix of w of length k, and there is no interval embedding of $\vec{U}$ into $w_1 \ldots w_{n-1}$. We set

$$\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t) = \sum_{w \in S^{\mathcal{P}}(\vec{U})} wt(w),$$

and given two sequences $\vec{U}$ and $\vec{V}$ of intervals of $\mathcal{P}$, we say that $\vec{U}$ is $\mathcal{P}$-Wilf equivalent to $\vec{V}$, written as $\vec{U} \sim_{\mathcal{P}} \vec{V}$, if $\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t) = \mathcal{S}^{\mathcal{P}}(\vec{V}, x, t)$.

We show that under mild assumptions on $\mathcal{P}$, $S^{\mathcal{P}}(\vec{U})$ is accepted by a finite automaton and, hence, $\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t)$ is a rational function. We compute $\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t)$ for various special cases of $\vec{U}$ and use these computations to establish various non-trivial Wilf-equivalences in this setting.

[1] S. Kitaev, J. Liese, J. Remmel, and B.E. Sagan, Rationality of generalized containments in words and Wilf equivalence, *Electron. J. Combin.*, **16(2)** (2009), R22

[2] T. Langley, J. Liese, and J. Remmel, Generating functions for Wilf equivalence under the generalized factor order, *J. Integer Seq.*, **14** (2011), 11.4.2

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