Generating functions for permutations with no consecutive pattern matches within the cycles

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Given a sequence $\sigma = \sigma_1 \dots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Let Υ be a set of permutations and let σ be a permutation in S_n with k cycles $C_1 \dots C_k$. Then we say that σ has a *cycle* Υ*-match* (c-Υ-match) if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ and an r such that $\text{red}(c_{r,i} \dots c_{r+j-1,i}) \in \Upsilon$ where we take indices of the form $r+s$ modulo p_i. Let $\mathcal{NCM}_n(\Upsilon)$ be the set of all permutations $\sigma \in S_n$ such that σ has no cycle Υ-match. We have been able to get closed form generating functions of the following form for certain sets of patterns Υ.

$$NCM_{\Upsilon}(t) = \sum_{n\geq 0} \frac{t^n}{n!} |\mathcal{NCM}_n(\Upsilon)|$$

Results: Let Γ be the set of all permutations $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 \in S_5$ such that

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5.$$

Let $\Upsilon_1 = \Gamma \cup \{1234\}$ then

$$NCM_{\Upsilon_1}(t) = \frac{2e^{t^2/2}e^{t^4/12}}{2 - 2t + t^2e^{-t}}.$$

Let $\Upsilon_2 = \Gamma \cup \{132, 1234\} = \{132, 1234, 35241, 45231, 34251\}$ then

$$NCM_{\Upsilon_2}(t) = \frac{2e^t e^{t^2/2}}{4 - 2e^t + t^2 + 2t}.$$

Let $\Upsilon_3 = \Gamma \cup \{231, 1234\} = \{231, 1234, 13254, 14253, 15243\}$ then

$$NCM_{\Upsilon_3}(t) = \frac{e^t e^{t^2/2}}{-1 - t + 2e^t - te^t}.$$