Bounds for the number of permutations containing a low density of patterns

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We seek to find a result similar to the Stanley-Wilf conjecture, but for permutations containing a low density of a certain pattern. In our case, instead of an exponential bound, we find a bound that is exponentially suppressed, and show that such a bound is indeed necessary.

Previous work in pattern avoidance has focused on finding bounds for the number of permutations with no occurrences of a given pattern. Let $S_n(\gamma)$ be the number of permutations in S_n that avoid the pattern γ. For any pattern of length three, $S_n(\gamma) = C_n$, the n-th Catalan number. In general, it is not the case that $S_n(\gamma)$ only depends of the length of γ. Bóna showed that $S_n(1234) < S_n(1324)$, and these are the only other patterns with known formulae. The most sweeping result concerning bounds for the number of pattern avoiding permutations is the Stanley-Wilf conjecture, recently proved by Marcus and Tardos.

Theorem 1 (Stanley-Wilf conjecture, 1980)**.** *Let γ be any pattern. Then there exists a constant c so that for all positive integers, we have $S_n(\gamma) \le c^n$.*

Let γ be a pattern of length f, and let $\chi_\delta^n(\gamma)$ be the number of permutations in S_n with fewer than $\delta^f n^f$ f-patterns. Our goal is to prove the following theorem:

Theorem 2. *For every f, $\delta < 1/(2f)$, there are N, a, b, such that for $n > N$, we have*

$$(a^n)n! \le \chi_\delta^n(\gamma) \le (b^n)n!$$

In particular, we have $a = \delta^f/2$, and $b = \left(\frac{e}{(f-1)\delta}\right)^\delta \left(\frac{f-1}{f}\right)^{1/f} + t$ for any $t > 0$.

Note that the bound we have is indeed non trivial. The term $\left(\frac{e}{(f-1)\delta}\right)^\delta$ approaches 1, as $\delta \to 0$, and the second term $\left(\frac{f-1}{f}\right)^{1/f}$ is constant smaller than 1, depending only on the length of the permutation γ. Furthermore, we can choose t small enough, so that the quantity

$$\left(\frac{e}{(f-1)\delta}\right)^\delta \left(\frac{f-1}{f}\right)^{1/f} + t$$

is strictly smaller than 1, giving a nontrivial bound. In particular, for a pattern of length 3, the theorem above, with choice of $\delta = .001$ and $t = .01$ implies that for n sufficiently large, there are at most

$$n! \left\{ \left(\frac{e}{(2).001}\right)^{.001} \left(\frac{2}{3}\right)^{1/3} + .01 \right\}^n < n!(.9)^n$$

patterns with fewer than $.001n^3$ 132-patterns.