Adin-Roichman-Mansour type identities

Sen-Peng Eu (National University of Kaohsiung, and Air Force Academy)

In [**?**], Adin and Roichman proved analytically the following identities, where $\mathsf{ldes}(\pi)$ denotes the position of the last descent. At the same time, Mansour [**?**] found a variation for $\mathfrak{S}_n(132)$.

Theorem 0.1 (Adin-Roichman)**.** *Let $\mathfrak{S}_n(321)$ be the set of 321-avoiding permutations in $\mathfrak{S}_n$. The following identities hold.*

$$\sum_{\pi\in\mathfrak{S}_{2n+1}(321)} (-1)^{\mathit{inv}(\pi)} q^{\mathit{ldes}(\pi)} = \sum_{\pi\in\mathfrak{S}_n(321)} q^{2\cdot \mathit{ldes}(\pi)}, \qquad \text{for } n \geq 0,$$

$$\sum_{\pi\in\mathfrak{S}_{2n}(321)} (-1)^{\mathit{inv}(\pi)} q^{\mathit{ldes}(\pi)} = (1-q) \sum_{\pi\in\mathfrak{S}_n(321)} q^{2\cdot \mathit{ldes}(\pi)}, \qquad \text{for } n \geq 1.$$

Exhausting computer research shows that this "$2n$ reduces to n" phenomenon is indeed rare. In this work, we would like to give several new A-R-M type identities, e.g:

Theorem 0.2. *Let $\mathfrak{B}_n(321)$ be the set of 321-avoiding Baxter permutations in $\mathfrak{S}_n$. For $n \geq 0$, we have*

$$\sum_{\pi\in\mathfrak{B}_{2n+1}(321)} (-1)^{\mathrm{maj}(\pi)} p^{\mathit{fix}(\pi)} q^{\mathrm{des}(\pi)} = p \cdot \sum_{\pi\in\mathfrak{B}_n(321)} p^{2\cdot \mathit{fix}(\pi)} q^{2\cdot\mathrm{des}(\pi)}.$$

Theorem 0.3. *Let $\mathit{Alt}_n(321)$ be the set of 321-avoiding alternating permutations in $\mathfrak{S}_n$, and let $\mathit{lead}(\pi) = \pi_1$, the first entry of π. For all $n \geq 1$, we have*

(i) $$\sum_{\pi\in \mathit{Alt}_{4n+2}(321)} (-1)^{\mathit{inv}(\pi)} \cdot q^{\mathit{lead}(\pi)} = (-1)^{n+1} \sum_{\pi\in \mathit{Alt}_{2n}(321)} q^{2\cdot \mathit{lead}(\pi)}$$

(ii) $$\sum_{\pi\in \mathit{Alt}_{4n+1}(321)} (-1)^{\mathit{inv}(\pi)} \cdot q^{\mathit{lead}(\pi)} = (-1)^{n} \sum_{\pi\in \mathit{Alt}_{2n}(321)} q^{2\cdot \mathit{lead}(\pi)}$$

(iii) $$\sum_{\pi\in \mathit{Alt}_{4n}(321)} (-1)^{\mathit{inv}(\pi)} \cdot q^{\mathit{lead}(\pi)} = (-1)^{n+1}(1-q) \sum_{\pi\in \mathit{Alt}_{2n}(321)} q^{2(\mathit{lead}(\pi)-1)}$$

(iv) $$\sum_{\pi\in \mathit{Alt}_{4n-1}(321)} (-1)^{\mathit{inv}(\pi)} \cdot q^{\mathit{lead}(\pi)} = (-1)^{n}(1-q) \sum_{\pi\in \mathit{Alt}_{2n}(321)} q^{2(\mathit{lead}(\pi)-1)}.$$

Theorem 0.4. *Let $\mathcal{DS}_n(312)$ be the set of 312-avoiding double simsum permutations in $\mathfrak{S}_n$, then*

(i) $$\sum_{\pi\in \mathcal{DS}_{2n+2}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathit{fix}(\pi)} = (-1+q^2) \sum_{\pi\in \mathcal{DS}_n(312)} q^{2\mathit{fix}(\pi)}, \text{ for } n \geq 1.$$

(ii) $$\sum_{\pi\in \mathcal{DS}_{2n-1}(312)} (-1)^{\mathrm{maj}(\pi)} \cdot q^{\mathit{lead}(\pi)} = \frac{2}{q(1+q^2)} \sum_{\pi\in \mathcal{DS}_n(312)} q^{2\mathit{lead}(\pi)}, \text{ for } n \geq 2.$$

These results are co-worked with T.S Fu, Y.J. Pan and P.L. Yan.

[1] R.M. Adin, Y. Roichman, Equidistribution and sign-balance on 321-avoiding permutations, Sémin. Loth. Combin. 51 (2004) B51d. ArXiv:math.CO/0304429.

[2] T. Mansour, Equidistribution and sign-balance on 132-avoiding permutations, Séminaire Lotharingien de Combinatoire 51 (2004) B51e.