Consecutive Patterns in up-down permutations

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Let A_n denote the set of up-down permutations of length n. For any sequence of distinct integers $\sigma_1, \ldots, \sigma_n$, we define $\mathrm{red}(\sigma)$ to be the permutation that results by replacing the i-th smallest integer in σ by i. If $\tau \in A_{2j}$, then we say that an up-down permutation $\sigma = \sigma_1 \ldots \sigma_n \in A_n$ has a τ-match at position i if $\mathrm{red}(\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+2j-1}) = \tau$ and we define $\tau\text{-mch}(\sigma)$ to be the number of τ-matches in σ. We say that $\tau \in A_{2j}$ has the *alternating minimal overlapping property* if two τ-matches in an alternating permutation $\sigma \in A_n$ can share at most two letters. For such a τ, we say that $\sigma \in A_{m(2j-2)+2}$ is an *maximal packing* for τ if $\tau\text{-mch}(\sigma) = m$, i.e., σ has the maximum number of possible τ-matches.

Let τ be an up-down permutation of length $2j$ with the alternating minimal overlapping property. We define the *generalized maximum packing polynomial of* τ $GMP_{\tau,2n}(x)$ as follows. Let $\mathcal{L}$ be the set of compositions $\alpha = (2a_1, 2a_2, 2a_3, \ldots, 2a_\ell)$ of $2n$ such that $a_1 \geq 0$, $a_i > 0$ for all $1 < i \leq \ell$, and $a_i = 1 \bmod j-1$ for all even i. Suppose that $\alpha = (2a_1, 2a_2, 2a_3, \ldots, 2a_\ell) \in \mathcal{L}$. Then we let $gmp_\tau(\alpha)$ be the number of permutations $\sigma = \sigma_1 \ldots \sigma_{2n}$ such that if we decompose σ into sequences as $\sigma = \sigma^{(1)} \ldots \sigma^{(\ell)}$ where $\sigma^{(i)}$ has length $2a_i$ for $i = 1, \ldots, \ell$, then (i) $\sigma^{(i)}$ is an increasing sequence if i is odd, (ii) $\mathrm{red}(\sigma^{(i)})$ is a maximum packing for τ if i is even, and (iii) the last element of $\sigma^{(i)}$ is less than the first element of $\sigma^{(i+1)}$ for $i = 1, \ldots, \ell-1$. We define the weight of the composition α to be $wt(\alpha) = gmp_\tau(\alpha)(-1)^{a_1\chi(a_1>0)}(-1)^{\sum_{s\geq 2}(a_{2s-1})}(x-1)^{\sum_{s\geq 1}\frac{2a_{2s}-2}{2j-2}}$ where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. We define $GMP_{\tau,2n}(x) = \sum_{\alpha\in\mathcal{L}} wt(\alpha)$.

Duane and Remmel proved that for any $\tau \in A_{2j}$ with the alternating minimal overlapping property,

$$1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in A_{2n}} x^{\tau-mch(\sigma)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} GMP_{\tau,2n}(x)}.$$

Thus in order to be able to explicitly calculate this generating function, we need to be able to compute $GMP_{\tau,2n}(x)$. In this paper, we focus on the problem of computing $GMP_{\tau,2n}(x)$. We will describe several infinite families of up-down permutations τ with the alternating minimally overlapping property for which $GMP_{\tau,2n}(x)$ can be computed via simple recursions. In such situations, we can compute the generating function $1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in A_{2n}} x^{\tau-mch(\sigma)}$.

This is joint work with Jeffrey Remmel.