Enumeration of permutations sorted with two passes through a stack and D_8 symmetries

Mathilde Bouvel (LaBRI, CNRS, Univ. Bordeaux)

We denote by $\mathbf{S}$ the stack sorting operator on permutations, and by D_8 the eight element group generated by the usual transforms $\mathbf{r}$ (reverse), $\mathbf{c}$ (complement) and $\mathbf{i}$ (inverse). We study the set of permutations that are sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$ (denoted $\mathrm{Id}(\mathbf{S} \circ \alpha \circ \mathbf{S})$) for $\alpha \in D_8$. We provide a characterization by (generalized) excluded patterns and enumeration results, that are refined according to a number of usual statistics on permutations.

Theorem 0.1. *The sets of permutations that are sorted by $S \circ \alpha \circ S$, for any α in D_8 are:*

(i) $\mathrm{Id}(\mathbf{S} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{r} \circ \mathbf{S}) = Av(2341, 3\bar{5}241)$*;*

(ii) $\mathrm{Id}(\mathbf{S} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{r} \circ \mathbf{S}) = Av(231)$*;*

(iii) $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{S}) = Av(1342, 31\text{-}4\text{-}2) = Av(1342, 3\bar{5}142)$*;*

(iv) $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S}) = Av(3412, 3\text{-}4\text{-}21)$.

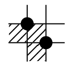
As we know since the seminal work of Knuth, the set $Av(231)$ (of one-stack sortable permutations) is enumerated by the Catalan numbers $Cat_n = \frac{1}{n+1}\binom{2n}{n}$. West has conjectured the set $Av(2341, 3\bar{5}241)$ of two-stack sortable permutations is enumerated by $\frac{2(3n)!}{(n+1)!(2n+1)!}$, and this formula has been proved by Dulucq, Gire and Guibert. For the two other sets, conjectures on their enumeration (refined with the distribution of some statistics) have been proposed by Claesson, Dukes and Steingrímsson. We prove these conjectures, that are stated as Theorems **??** and **??** below.

Theorem 0.2. *The two sets* $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ *and* $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$ *are enumerated according to the size of the permutations by the same sequence. Moreover, the tuple of statistics* $(\mathrm{updownword}, \mathrm{rmax}, \mathrm{lmax}, \mathrm{zeil}, \mathrm{indmax}, \mathrm{slmax}, \mathrm{slmax} \circ \mathbf{r})$ *has the same distribution on both sets.*

The updownword statistics associates a word $w \in \{u, d\}^{n-1}$ to each permutation σ of size n, with $w_i = u$ (resp. d) if $\sigma(i) < \sigma(i+1)$ (resp. $\sigma(i) > \sigma(i+1)$). The equidistribution of the statistics updownword implies that the following statistics are also equidistributed in $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ and $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$: $(\mathrm{des}, \mathrm{maj}, \mathrm{maj} \circ \mathbf{r}, \mathrm{maj} \circ \mathbf{c}, \mathrm{maj} \circ \mathbf{rc}, \mathrm{valley}, \mathrm{peak}, \mathrm{ddes}, \mathrm{dasc}, \mathrm{rir}, \mathrm{rdr}, \mathrm{lir}, \mathrm{ldr})$. Consequently, the bijection of Theorem **??** preserves the *joint* distribution of a 20-tuple of statistics on permutations.

Theorem 0.3. *The set* $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ *is enumerated by the Baxter numbers*

$$Bax_n = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

Moreover, the triple of statistics $(\mathrm{des}, \mathrm{lmax}, \mathrm{comp})$ *has the same distribution on* $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ *and on the set* $Av(2\text{-}41\text{-}3, 3\text{-}14\text{-}2)$ *of Baxter permutations. It also has the same distribution than the triple of statistics* $(\mathrm{lmax}, \mathrm{occ}_\mu, \mathrm{comp})$ *on the set* $Av(2\text{-}41\text{-}3, 3\text{-}41\text{-}2)$ *of twisted Baxter permutations, where* occ_μ *denoted the number of occurrences of the mesh pattern* $\mu =$  .

Theorems **??** and **??** are proved using generating trees and rewriting systems. Furthermore, the proof of Theorem **??** makes use of a recent effective bijection of Giraudo between Baxter permutations, twisted Baxter permutations and pairs of twin binary trees.

This is joint work with Olivier Guibert.