Periodic patterns of k-shifts

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If $f : A \to A$, where A is a linearly ordered set, we define the pattern of f at x (of length n) as $\text{Pat}(x, f, n) = \rho(x, f(x), f^2(x), \cdots, f^{n-1}(x)) \in S_n$ where ρ is a reduction map that takes in a list of different elements and returns a permutation π by labeling the smallest element in the list with a 1, the second smallest with a 2, and so on. For example, $\rho(3, 6, 2, 3.4, 100, -2) = 352461$.

For instance, if we define the *binary shift* Σ_2 on the set of infinite binary words with lexicographic order by

$$\Sigma_2(w_1 w_2 w_3 \cdots) = w_2 w_3 w_4 \cdots,$$

we have that $\text{Pat}(01101001\cdots, \Sigma_2, 5) = 25413$, since

$$01001\cdots < 01101001\cdots < 1001\cdots < 101001\cdots < 1101001\cdots.$$

The above definition assumes that the values $x, f(x), \ldots, f^{n-1}(x)$ are all different. Another interesting case is when $x \in A$ is a n-periodic point of f, that is, $f^n(x) = x$ but $f^i(x) \neq x$ for $1 \leq i < n$. In this case, we say that $[\pi]$ is the *periodic pattern* of f at x if $\text{Pat}(x, f, n) = \pi$. The n-periodic points of the binary shift are exactly those sequences $w = (w_1 w_2 \cdots w_n)^\infty$ where the word $w_1 w_2 \cdots w_n$ is primitive (that is, there is no $k > 1$ so that $w_1 w_2 \cdots w_n = v^k$). For example, the periodic pattern of Σ_2 at $(00101)^\infty$ is $[13524]$.

It was shown in [**?**] that for any given piecewise monotone function $f : I \to I$, where $I \in \mathbb{R}$ is a closed interval, there exist patterns π that are not realized by f, that is, there is no $x \in I$ with $\text{Pat}(x, f, n) = \pi$.

In [**?**], Elizalde characterized and enumerated the patterns realized by the k-shift Σ_k (i.e., the shift on k-ary words), which is equivalent to the map $x \mapsto kx \bmod 1$ on the unit interval.

In this talk, we describe and enumerate the *periodic* patterns of a few maps, including the k-shift map and the tent map, which is defined on $[0, 1]$ as

$$x \mapsto \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & 1/2 < x \leq 1. \end{cases}$$

As a byproduct of the enumeration of periodic patterns of Σ_k, we derive a recursive formula describing the number of cyclic permutations of length n with $k - 1$ descents:

$$C(n, k) = L_k(n) - \sum_{i=2}^{k-1} \binom{n + k - i}{k - i} C(n, i),$$

where

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu(d) k^{\frac{n}{d}}$$

is the number of length n primitive words on k letters. The number of length n periodic patterns of the k-shift is then $\sum_{i=2}^{n} C(n, i)$.

We also study periodic patterns of the reverse k-shift, which is equivalent to the map $x \mapsto 1 - kx \bmod 1$ on the unit interval. For this purpose, we define an order on infinite sequences on k letters that plays the role that the lexicographic order has for the k-shift. We conjecture that the number of length n periodic patterns of the reverse k-shift is the same as the number of length n periodic patterns of the k-shift, but its relationship to the number of cyclic permutations of length n with $k - 1$ ascents is somewhat more complicated in the cases when n is two times an odd number.

This is joint work with Sergi Elizalde.

[1] C. Bandt, G. Keller and B. Pompe, Entropy of interval maps via permutations, *Nonlinearity* 15 (2002), 1595–1602.

[2] S. Elizalde, The number of permutations realized by a shift, *SIAM J. Discrete Math.* 23 (2009), 765–786.