Thresholds for patterns in random permutations

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(This talk is based on joint work with David Bevan.)

In this talk we will investigate thresholds for the appearance and disappearance of consecutive patterns occurring within large random permutations as the number of inversions increases. Let $\boldsymbol{\sigma}_{n,m}$ denote a permutation chosen uniformly from the set of n-permutations with exactly m inversions. We call $\boldsymbol{\sigma}_{n,m}$ the *uniform random permutation.*

As the number of inversions increases, consecutive patterns appear before eventually disappearing in the following way: if π is any non-monotonic consecutive pattern, then there exist functions $f_\pi^-(n)$ and $f_\pi^+(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m} \text{ contains } \pi\big] = \begin{cases} 0 & \text{if } m \ll f_\pi^-(n) \\ 1 & \text{if } f_\pi^-(n) \ll m \text{ and } f_\pi^+(n) \ll \binom{n}{2} - m \\ 0 & \text{if } \binom{n}{2} - m \ll f_\pi^+(n) \end{cases}$$

where $f(n) \ll g(n)$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. We establish these lower and upper thresholds for any fixed consecutive pattern π. We also consider thresholds for classical patterns.

To do so, we work with inversion sequences, which we consider to be weak integer compositions. As a result, we introduce the following model of random integer compositions. Let $\mathcal{C}_{n,m}$ denote the set of all compositions of length n such that all terms sum to m. Let $p \in [0,1)$ and $q = 1-p$, then for each $m \geq 0$, we assign to each composition $C \in \mathcal{C}_{n,m}$ the probability $p^m q^n$. Each term is sampled independently from the geometric distribution with parameter q; that is, $\mathbb{P}\big[C(i) = k\big] = p^k q$ for each $k \geq 0$ and $i \in [n]$. We call such a random composition a *geometric random composition.* We establish that, asymptotically with probability tending to 1, a geometric random composition is an inversion sequence if and only if $p \ll 1$.

This talk will focus on how we transfer thresholds for patterns in the geometric random composition to get thresholds for patterns in the uniform random permutation.