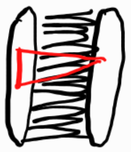
Transversal Embedding
(with Y. Cheng)

Scottish Combinatorics Meeting

May '23

Graph embedding: when does a host graph G contain a given subgraph H?



Computationally difficult – seek (easy to check) sufficient conditions.

Transversal embedding: $s \geq e(H)$ graphs on the same vertex set

$G_1 \; G_2 \; \dots \; G_s$

Is there a copy of H with ≤ 1 edge from each graph? Copy is transversal or rainbow.

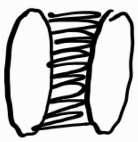
Generalisation: if every G_c is the same then recover uncoloured problem.

Examples

Edges vs. triangles

not colour-blind

Mantel 1907: Every n-vx graph with $> \lfloor \frac{n^2}{4} \rfloor$ edges contains a triangle

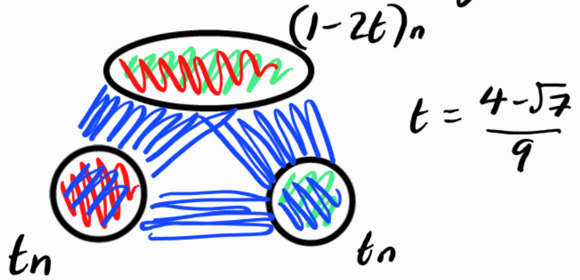


Aharoni–DeVos–de la Maza–Montejano–Šámal '20

G_1, G_2, G_3 n-vx graphs,

$e(G_i) > \left(\frac{26 - 2\sqrt{7}}{81} \right) n^2 \simeq 0.2557 n^2$

contain a rainbow triangle



Min degree vs Hamilton cycle

colour-blind

Dirac 1952: Every n-vx graph with min degree $\geq \frac{n}{2}$ contains a Hamilton cycle.

Joos–Kim '18:

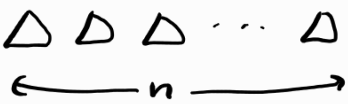
$G_1, \dots, G_n$ n-vx graphs,

$\delta(G_i) \geq \frac{n}{2}$ for all i

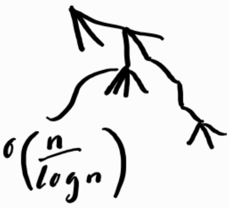
contain a rainbow Hamilton cycle.

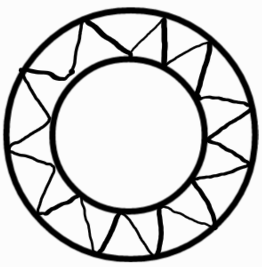
(Joos-Kim) Which embedding problems are colour-blind?

Min degree vs spanning H approximately colour-blind: $+\varepsilon n$ degree



Montgomery-Müyesser-Pehova '21





Gupta-Hamann-Müyesser-Parczyk-Sgueglia '22+

bounded degree
small bandwidth

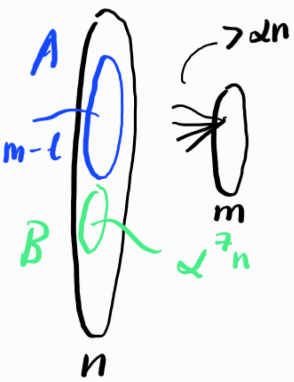
Chakraborti-Im-Kim-Liu '23+

Proof techniques (MMP)

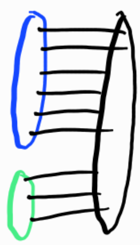
Absorbing: from almost finished to finished.

△ △ △ … △ n colours

Key lemma $m = \alpha^2 n$ $\ell = \alpha^9 n$



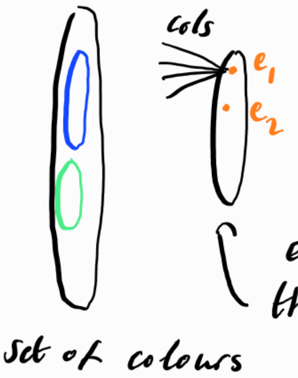
For any $B_0 \subset B$ of size ℓ there is a perfect matching from $A \cup B_0$.



Find absorber.

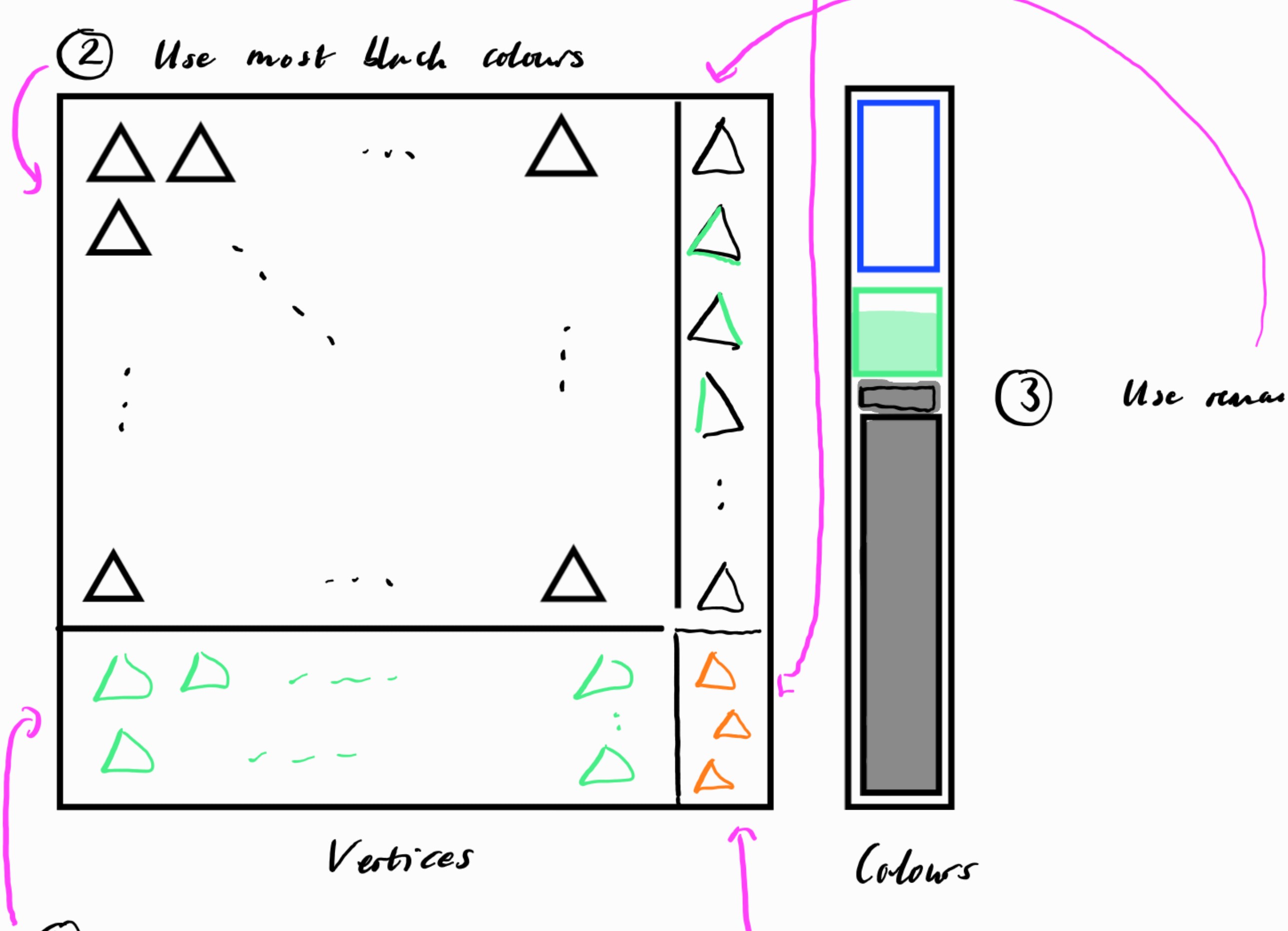
Many edges in thick graph (edges of many colours).

1. Find triangles in thick graph.



edges of triangles in thick graph

2. Use most black colours



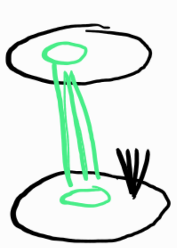
3. Use remaining black cols + green

4. Use remaining vertices (green cols)

5. Use the lemma to assign remaining colours

Blow-up lemma (Komlós - Sárközy - Szemerédi)

(+ regularity lemma) is a very useful tool for embedding spanning graphs

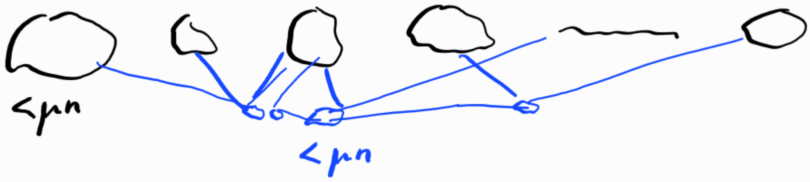


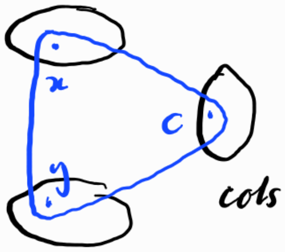
'quasirandom' : every not-too-small pair spans roughly the same proportion of edges as globally

reasonable min degree

$\longrightarrow$ contains any bounded degree graph 'that fits'

(Cheng - S.) Transversal blow-up lemma for separable graphs
bounded degree





'quasirandom'
reasonable min degree

$\longrightarrow$ contains any bounded degree separable graph 'that fits'

Use TBUL in place of BUL to get transversal versions of embedding results.

Open questions

$\binom{r}{2}$ graphs G_i, $e(G_i) > ?$ $\Rightarrow$ rainbow K_r?

Spanning: exact results $(n_0 + \varepsilon n)$

Other colour patterns between monochromatic and rainbow