Scottish Combinatorics Meeting

University of Strathclyde

22–23 May 2023

Monday 22 May

0930–1030 coffee / tea

1030–1035 welcome

1035–1125 **Ruth Hoffmann** (University of St Andrews)
Between Subgraph Isomorphism *and* Maximum Common Subgraph: *How to make faster algorithms*

1125–1215 **Akshay Gupte** (University of Edinburgh)
Submodular maximization over easy knapsack constraints

1215–1330 lunch

1330–1420 **Ciaran McCreesh** (University of Glasgow)
Is your combinatorial search algorithm telling the truth?

1420–1510 **Torsten Mütze** (University of Warwick)
Combinatorial generation: graphs, algorithms, polytopes, and optimization

1510–1545 tea / coffee

1545–1610 **Emma Smith** (Royal Holloway University of London)
A combinatorics and crypt applications sandwich:
Distinct difference configurations for wireless sensor networks

1610–1635 **Gemma Crowe** (Heriot-Watt University)
Conjugacy, languages and groups

1635–1700 **Dan Threlfall** (University of Strathclyde)
Thresholds for patterns in random permutations

1700–1725 **Namrata** (University of Warwick)
Kneser graphs are Hamiltonian

1845 dinner
The Italian Kitchen

Tuesday 23 May

0930–1020	**Jess Enright** (University of Glasgow) *Cops-and-robbers on multilayer graphs*
1020–1100	coffee / tea
1100–1150	**Katherine Staden** (The Open University) *Transversal embeddings*
1150–1215	**Bishal Deb** (University College London) *Laguerre digraphs and continued fractions*
1215–1330	lunch
1330–1420	**Simon Blackburn** (Royal Holloway University of London) *Permutations that separate close elements*
1420–1510	**Robert Brignall** (The Open University) *Well-quasi-ordering permutations*
1510–1545	tea / coffee
1545–1610	**Laura Johnson** (University of St Andrews) *Applications of partial difference families to partial designs*
1610–1700	**Natasha Blitvić** (Queen Mary University of London) *Combinatorial moment sequences*

Permutations that separate close elements

Simon R. Blackburn

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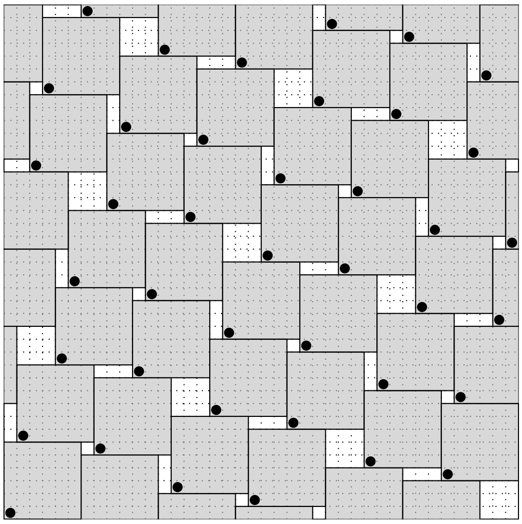
Royal Holloway University of London

(This talk is based on joint work with Tuvi Etzion.)

Look at the diagram below. It is drawn on an $n \times n$ torus (here with $n = 40$). There are n non-overlapping 6×6 squares; more generally, we will consider rectangles that are s cells wide and k cells high. The dots in the lower left-hand corners form a permutation: there is one dot in each row and each column. For fixed n and k, what is the largest value $\sigma(k, n)$ of s where such a construction is possible?



I proved (*J.Combin. Theory Ser. A*, 2023) that $\sigma(n, k)$ can only take one of two values: $\sigma(n, k) \in \{\lfloor (n-1)/k \rfloor - 1, \lfloor (n-1)/k \rfloor\}$. This establishes a conjecture of Mammoliti and Simpson from 2020. Tuvi Etzion and I have recently shown which of these two values $\sigma(n, k)$ takes, determining the value of $\sigma(n, k)$ for all values of n and k. In this talk, I will discuss these results and some of the techniques we use.

Combinatorial moment sequences

Natasha Blitvić

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Queen Mary University of London

(This talk is partly based on joint work [1, 2] with Einar Steingrímsson and Slim Kammoun.)

Take your favorite integer sequence. Is this sequence a sequence of moments of some probability measure on the real line? We will look at a number of interesting examples (some proven, others merely conjectured) of moment sequences in combinatorics. We will consider ways in which this positivity may be expected (or surprising!), the methods of proving it, and the consequences of having it. The problems we will consider will be very simple to formulate, but will take us up to the very edge of current knowledge in combinatorics, 'classical' probability, and noncommutative probability.

[1] N. Blitvić and E. Steingrímsson, Permutations, Moments, Measures. *Transactions of the American Mathematical Society*, Vol. 374, Number 8, August 2021, pp. 5473–5509.

[2] N. Blitvić, S. M. Kammoun, E. Steingrímsson. A new perspective on positivity in (consecutive) permutation patterns. *Proceedings of the FPSAC 2023*, July 17-21, Davis CA.

Well-quasi-ordering permutations

Robert Brignall

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(This talk is based on joint work with Vincent Vatter.)

The study of well-quasi-ordering in combinatorics includes some of the most celebrated results of the past 70 years, including Higman's Theorem, Kruskal's Tree Theorem and Robertson & Seymour's Graph Minor Theorem. The general set-up is to consider a family of combinatorial structures (such as graphs or permutations), and some form of ordering on this family (typically an embedding of smaller structures into larger ones, such as graph minor, induced subgraph, or permutation containment). Such an ordered family is *well-quasi-ordered* (wqo) if it contains no infinite antichains – that is, an infinite set of structures no two of which are comparable in the ordering.

The combinatorial structure of choice for most of this talk is the permutation, equipped with containment (which is the natural 'induced substructure' order). We know of an abundance of different infinite antichains of permutations (so the set of all permutations is certainly not wqo), but mostly researchers are interested in sets (or *classes*) which comprise the permutations that *avoid* some given set. Here we can ask: is a given class wqo?

This talk will survey recent developments in wqo for permutations, and the relationship with other notions of interest (such as the enumeration of permutation classes). A class being wqo is often seen as an indicator that the class is 'tame' (whereas those with infinite antichains are 'wild'), although a recent result (involving uncountably many different wqo classes of permutations) suggests that wqo classes are not as tame as we might have hoped. On the other hand, a stronger notion, known as *labelled* well-quasi-ordering, perhaps offers a better guarantee of 'tameness'.

Conjugacy, languages and groups

Gemma Crowe

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Heriot-Watt University

Formal language theory has found surprising applications in group theory in recent years. In this talk, we will introduce languages associated to conjugacy classes. By studying properties of these languages, we will explore what this tells us about counting conjugacy classes in a group. Finally, we will consider if and when certain properties of these languages can be extended via quasi-isometries, including recent work on virtual right-angled Artin groups.

Laguerre digraphs and continued fractions

Bishal Deb

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University College London

(This talk is based on joint work with Alexander Dyachenko, Matthias Pétréolle, Alan Sokal.)

A *Laguerre digraph* is a directed graph on n vertices such that each vertex has indegree 0 or 1 and outdegree 0 or 1. Thus, the connected components are either directed cycles or directed paths. They are a combinatorial interpretation of the coefficients of the Laguerre polynomials. A Laguerre digraph with no directed paths and only directed cycles is simply the digraph of a permutation in cycle notation. We will begin by introducing Laguerre digraphs.

We then introduce Flajolet's combinatorial theory of continued fractions and state a continued fraction identity for the series $\sum_{n=0}^{\infty} n!$ due to Euler (1760). There are several proofs known for this identity; our work focuses on two bijective proofs due to Foata–Zeilberger (1990) and Biane (1993).

In a series of recent papers with Dyachenko, Pétréolle and Sokal, we have analysed the intermediate steps in the Foata–Zeilberger and Biane bijections. The "Biane history" involves building up a Laguerre digraph by inserting vertices at each stage, whereas the "Foata–Zeilberger history" involves building up a Laguerre digraph by inserting edges at each stage. We will show simple examples to illustrate both histories without going into any technical details.

In [1], we solved conjectured continued fractions due to Randrianarivony–Zeng (1996), Sokal–Zeng (2022), and Deb–Sokal (arxiv:2022) using the Foata–Zeilberger history. In [3, 2], we extended various permutation statistics to Laguerre digraphs and used them along with the Biane bijection to combinatorially interpret the Stieltjes–Rogers matrices of Sokal–Zeng's second continued fraction for permutations. This approach also partly helps in solving a conjecture of Corteel–Sokal (2017) on the Hankel total-positivity of the Laguerre polynomials. We will end the talk by quickly stating some of these results.

No prerequisites will be required for this talk.

[1] B. Deb, Continued fractions using a Laguerre digraph interpretation of the Foata–Zeilberger bijection and its variants, in preparation.

[2] B. Deb, A. Dyachenko, M. Pétréolle and A.D. Sokal, Lattice paths and branched continued fractions III: Generalizations of the Laguerre, rook and Lah polynomials, in preparation.

[3] B. Deb, A.D. Sokal, Continued fractions for cycle-alternating permutations, arXiv preprint: https://arxiv.org/abs/2304.06545

Cops-and-robbers on multilayer graphs

Jess Enright

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University of Glasgow

(This talk is based on joint work with Kitty Meeks, Will Pettersen and John Sylvester.)

I will describe the game of cops-and-robbers and then its generalisation to multilayer graphs. In this setting, a graph consists of a single set of vertices with multiple (potentially intersecting) edge sets. We allow the cops and robber to move only on their assigned layer, and ask if the cops can be guaranteed to catch the robber in finite time. Using several examples, I'll show that initial intuition about the best way to allocate cops to layers is not always correct. I will outline arguments showing that the number of cops required to catch a robber in a multilayer graph is neither bounded from above nor below by any function of the cop numbers of the individual layers. Additionally, we'll talk about a question of worst-case division of a simple graph into layers: given a simple graph G, what is the maximum number of cops required to catch a robber over all multilayer graphs where each edge of G is in at least one layer and all layers are connected? For cliques, suitably dense random graphs, and graphs of bounded treewidth, we have determined this parameter up to multiplicative constants. Lastly I'll outline a multilayer variant of Meyniel's conjecture, and show the existence of an infinite family of graphs whose multiayer cop number is bounded from below by a constant times $n/\log n$, where n is the number of vertices in the graph.

Submodular maximization over easy knapsack constraints

Akshay Gupte

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A submodular function over the $\{0, 1\}$ lattice is regarded as a discrete analog of a convex function, and appears commonly in many combinatorial optimization problems. It can be minimised in polynomial-time, but maximisation is NP-hard although it is 1/2-approximable. Many approximation algorithms are known for maximising over different classes of independence families, such as constant number of matroids and $\leq$-knapsacks. We consider the question over the intersection of an independence family with a collection of $\leq$- and $\geq$-knapsacks that satisfy a certain property that is related to integrality of their covering polytope and their clutters. The feasible points in these knapsacks can be obtained by monomial orderings of binary vectors. We show that when k (number of knapsacks) is bounded by a constant then the maximum can be approximated to the same factor as that for submodular maximisation over the independence family. We also give a lower bound on approximability by establishing that there does not exist a randomised algorithm with approximation factor roughly $\Omega(\sqrt{k}/2^{\log n})$. This is established by showing reducibility of a large class of cardinality maximization problems to a combinatorial question that we propose for ordering integers under different permutations.

Between Subgraph Isomorphism and Maximum Common Subgraph, How to make faster algorithms

Ruth Hoffmann

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University of St Andrews

(This talk is based on joint work with Mun See Chang, Ciaran McCreesh and Craig Reilly.)

The subgraph isomorphism problem looks at finding a small pattern graph inside a larger target graph. Whereas when a small pattern graph does not occur inside a larger target graph, we can ask how to find "as much of the pattern as possible" inside the target graph. This is known as the maximum common subgraph problem.

We will look at the two different types of subgraph problems (and their variations), as well as talk about a restricted alternative [1] which asks if all but k vertices from the pattern can be found in the target graph.

Finally, we will look at ongoing research into making the algorithms involved in these problems faster by using a combination of homomorphisms, and subgraph homomorphism search to inform the original problem as to where the pattern graph will never occur in the target graph.

[1] R. Hoffmann, C. McCreesh, C. Reilly. Between Subgraph Isomorphism and Maximum Common Subgraph *AAAI 2017*.

Applications of Partial Difference Families to Partial Designs

Laura Johnson

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(This talk is based on joint work with Dr Sophie Huczynska.)

In 1939 Bose and Nair defined partially balanced incomplete block designs (PBIBDs). These are the partial analogue of BIBDs. More recently, in the early 2000s, Ogata et al. defined a new combinatorial design known as a splitting balanced incomplete block design (splitting BIBD). The motivation for defining splitting BIBDs was to construct AMD codes, a type of cryptographical tool used to protect against attacks from active adversaries.

In this talk I will discuss how two recently introduced combinatorial structures known as disjoint partial difference families (DPDFs) and external partial difference families (EPDFs) may be used to find constructions of these designs.

Is Your Combinatorial Search Algorithm Telling the Truth?

Ciaran McCreesh

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University of Glasgow

How do you know whether your combinatorial search algorithm is implemented correctly? You could try testing it, but even if you're convinced you've done a thorough job, will anyone else believe you? Another possibility is "correct by construction" software created using formal methods—but these methods are far from being able to approach the complexity or performance of modern satisfiability or constraint programming solvers. In this talk I'll tell you about a third option, called *proof logging* or *certifying*. The idea is that, alongside a solution, an algorithm must produce a mathematical proof in a standard format that demonstrates that the solution is correct. This proof can be verified by an independent proof checker, which should be much simpler, and thus easier to trust. The key challenge in getting this to work is to find a proof language which is both simple to verify, and expressive enough to cover a wide range of solving techniques with very low overheads. It's not obvious that such a language should even exist, but I'll argue that cutting planes with a dominance-based extension rule might be exactly what we need: even though cutting planes has no notion of what vertices, graphs, or even integers are, it is strong enough to verify the reasoning used in state of the art algorithms for problems like subgraph isomorphism, clique, and maximum common connected subgraph, and even in constraint programming solvers.

Combinatorial generation: graphs, algorithms, polytopes, and optimization

Torsten Mütze

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University of Warwick

In mathematics and computer science we frequently encounter different classes of combinatorial objects. In this talk I focus on algorithms for efficiently generating these objects, i.e., an algorithm should visit each of the objects from the class exactly once. This problem has ramifications into algorithms, graph theory, algebra, geometry etc., which I will highlight in this talk. Moreover, I present two recent frameworks which allow solving the generation problem systematically for a large variety of different objects. The first framework is based on encoding the combinatorial objects by permutations. The second framework uses combinatorial optimization as a black box for the purpose of generation. The listings of objects computed by both frameworks correspond to Hamilton paths and cycles on very general classes of polytopes.

Kneser graphs are Hamiltonian

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(This talk is based on joint work with Arturo Merino and Torsten Mütze.)

For integers $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n,k)$ has as vertices all k-element subsets of an n-element ground set, and an edge between any two disjoint sets. It has been conjectured since the 1970s that all Kneser graphs admit a Hamilton cycle, with one notable exception, namely the Petersen graph $K(5,2)$. This problem received considerable attention in the literature, including a recent solution for the sparsest case $n = 2k+1$. The main contribution of this work is to prove the conjecture in full generality. We also extend this Hamiltonicity result to all connected generalized Johnson graphs (except the Petersen graph). The generalized Johnson graph $J(n,k,s)$ has as vertices all k-element subsets of an n-element ground set, and an edge between any two sets whose intersection has size exactly s. Clearly, we have $K(n,k) = J(n,k,0)$, i.e., generalized Johnson graph include Kneser graphs as a special case. Our results imply that all known families of vertex-transitive graphs defined by intersecting set systems have a Hamilton cycle, which settles an interesting special case of Lovász' conjecture on Hamilton cycles in vertex-transitive graphs from 1970. Our main technical innovation is to study cycles in Kneser graphs by a kinetic system of multiple gliders that move at different speeds and that interact over time, somewhat reminiscent of the gliders in Conway's Game of Life, and to analyze this system combinatorially and via linear algebra.

A combinatorics and crypt applications sandwich: Distinct Difference Configurations for Wireless Sensor Networks

Emma Smith

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Royal Holloway University of London

(This talk is based on joint work with Simon Blackburn and Luke Stewart.)

A Distinct Difference Configuration (DDC) is a subset of a group, whose pairwise differences are distinct. Wireless Sensor Networks (WSNs) are networks of spatially distributed devices that measure their environment and pass these measurements across the network. This talk will sandwich how DDCs can be used for key distribution in WSNs, between some background on DDCs themselves and some recent results inspired by the application.

Transversal embeddings

Katherine Staden

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(This talk is based on joint work with Yangyang Cheng.)

A classical question in graph theory is to find sufficient conditions which guarantee that a graph G contains a given spanning subgraph H. A colourful variant of this problem has graphs $G_1, \ldots, G_s$ on the same vertex set, where $s \geq e(H)$ and we think of each graph as having a different colour, and the goal is to find a *transversal* (or *rainbow*) copy of H that contains at most one edge from each graph G_i. I will survey this area and its proof techniques, and will discuss some joint work with Yangyang Cheng on regularity tools in this setting.

Thresholds for patterns in random permutations

Dan Threlfall

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(This talk is based on joint work with David Bevan.)

In this talk we will investigate thresholds for the appearance and disappearance of consecutive patterns occurring within large random permutations as the number of inversions increases. Let $\boldsymbol{\sigma}_{n,m}$ denote a permutation chosen uniformly from the set of n-permutations with exactly m inversions. We call $\boldsymbol{\sigma}_{n,m}$ the *uniform random permutation.*

As the number of inversions increases, consecutive patterns appear before eventually disappearing in the following way: if π is any non-monotonic consecutive pattern, then there exist functions $f_\pi^-(n)$ and $f_\pi^+(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m} \text{ contains } \pi\big] = \begin{cases} 0 & \text{if } m \ll f_\pi^-(n) \\ 1 & \text{if } f_\pi^-(n) \ll m \text{ and } f_\pi^+(n) \ll \binom{n}{2} - m \\ 0 & \text{if } \binom{n}{2} - m \ll f_\pi^+(n) \end{cases}$$

where $f(n) \ll g(n)$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. We establish these lower and upper thresholds for any fixed consecutive pattern π. We also consider thresholds for classical patterns.

To do so, we work with inversion sequences, which we consider to be weak integer compositions. As a result, we introduce the following model of random integer compositions. Let $\mathcal{C}_{n,m}$ denote the set of all compositions of length n such that all terms sum to m. Let $p \in [0, 1)$ and $q = 1-p$, then for each $m \geq 0$, we assign to each composition $C \in \mathcal{C}_{n,m}$ the probability $p^m q^n$. Each term is sampled independently from the geometric distribution with parameter q; that is, $\mathbb{P}\big[C(i) = k\big] = p^k q$ for each $k \geq 0$ and $i \in [n]$. We call such a random composition a *geometric random composition.* We establish that, asymptotically with probability tending to 1, a geometric random composition is an inversion sequence if and only if $p \ll 1$.

This talk will focus on how we transfer thresholds for patterns in the geometric random composition to get thresholds for patterns in the uniform random permutation.