Kneser graphs are Hamiltonian

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Scottish Combinatorics Meeting 2023
Introduction

- **Kneser graph** $K(n, k)$
  
  vertices: $k$-elements subsets of \{1, 2, ..., $n$\}
  
  edges: $(A, B)$, where $A \cap B = \emptyset$
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Complete graph $K(4, 1)$
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- assumption: $k \geq 1$ and $n \geq 2k + 1$
Origin

- *Conjecture ([Kneser 1955]):* If the $k$-subsets of a $n$-set are divided into $n - 2k + 1$ classes, then two disjoint subsets end up in the same class.
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  \[ \chi(K(n, k)) = n - 2k + 2 \]
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- special case: **Kneser graph**, with Petersen graph $K(5,2)$ as an exception
Hamiltonicity in dense Kneser graphs

- ‘dense’ if $n$ is large w.r.t. $k$
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• [Y. Chen+Füredi 2002]: short proof for $n = ck, c \in \{3,4,\ldots\}$
Hamiltonicity in sparse Kneser graphs

- sparsest case: $n = 2k + 1$
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- **open**: $2k + 3 \leq n \leq (1 + o(1)) 2.62 k$, where $n \neq 2k + 2^a$
Our 1st result

• Theorem 1 [STOC 2023]: $K(n, k)$ has a Hamilton cycle for $k \geq 1$ and $n \geq 2k + 1$, unless $(n, k) = (5, 2)$. 
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- settles Hamiltonicity of $K(n, k)$ in full generality
Generalized Johnson graphs

- generalized Johnson graphs $J(n, k, s)$
  
  vertices: $k$-elements subsets of $\{1, 2, \ldots, n\}$
  
  edges: pairs of sets $(A, B)$, where $|A \cap B| = s$
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Generalized Johnson graphs

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- vertex-transitive
Our 2nd result

- Theorem 2 [STOC 2023]: $J(n, k, s)$ has a Hamilton cycle for $k \geq 1$ and $n \geq 2k + 1$, unless $(n, k, s) = (5, 2, 0), (5, 3, 1)$, the Petersen graph.
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- settles the Hamiltonicity problem for graphs defined by intersecting set-systems
Proof outline

• construct a cycle factor: collection of cycles covering all vertices
  (it works for $n \geq 2k + 1$)
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• glue cycles together (the assumption $n \geq 2k + 3$ is important)
Cycle factors

- $k$-subset of $n$ is represented by a binary string of length $n$ with $k$ 1s.
Cycle factors

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- Example: $n = 11, k = 4, x = \{4, 7, 8, 10\}$
Cycle factors

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\end{array}
\]

\(x\)

\(\text{denote 1 by [ and 0 by ]}\)
Cycle factors

- denote 1 by [ and 0 by ]

- cyclical parenthesis matching (closest pairs of 1s and 0s)
Cycle factors

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Cycle factors

• $f$: complement the matched bits
Cycle factors

- \( f \): complement the matched bits

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{array}
\]
Cycle factors

- edge in a Kneser graph: \((x, f(x))\)

- \(f\): complement the matched bits
Cycle factors

- $f$: complement the matched bits
- edge in a Kneser graph: $(x, f(x))$
- repeated application of $f$ gives a cycle
Cycle factors

• complement the matched bits

\[ f(00010011010) = 10001000101 \]

- \( f \): complement the matched bits
- edge in a Kneser graph: \((x, f(x))\)
- repeated application of \( f \) gives a cycle
- partitions the vertices of \( K(n, k) \) into disjoint cycles
Example

$K(6, 2)$
Example

\[
\begin{array}{ccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\[f\]

\[K(6, 2)\]
Example

\[ f(6, 2) \]

\[ \{1, 4\} \]

\[ \{1, 5\} \]

\[ \{1, 6\} \]

\[ \{2, 3\} \]

\[ \{2, 4\} \]

\[ \{2, 5\} \]

\[ \{2, 6\} \]

\[ \{3, 4\} \]

\[ \{3, 5\} \]

\[ \{3, 6\} \]

\[ \{4, 5\} \]

\[ \{4, 6\} \]

\[ \{5, 6\} \]

\[ K(6, 2) \]
Example

\[
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & 0 \\
f \downarrow & & & & & \\
0 & 0 & 1 & 1 & 0 & 0 \\
f \downarrow & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

\[K(6, 2)\]
Example

\[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\]
Example

\[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

\(K(6, 2)\)
Example

\begin{itemize}
\item \[\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{array}\]
\end{itemize}

\[K(6, 2)\]
### Example

<table>
<thead>
<tr>
<th>1 0 1 0 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 0 1 0 0</td>
</tr>
<tr>
<td>0 0 1 0 1 0</td>
</tr>
<tr>
<td>0 0 0 1 0 1</td>
</tr>
<tr>
<td>1 0 0 0 1 0</td>
</tr>
</tbody>
</table>

Graph $K(6, 2)$

- Nodes: $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}$
- Edges: All possible edges between nodes.
Example

\[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[K(6, 2)\]
Example

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\( f \)

\( K(6, 2) \)
Example

\[ f(6, 2) \]

\[ K(6, 2) \]
Example

\[
\begin{array}{ccccccc}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\{1,4\} & \{1,5\} & \{1,6\} & \{2,3\} & \{2,4\} & \{2,5\} \\
\{2,6\} & \{3,4\} & \{3,5\} & \{3,6\} & \{4,5\} & \{4,6\} \\
\{5,6\} & \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} \\
\end{array}
\]
Example

\[
\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

\[K(6, 2)\]
Example

\[ f(6, 2) \]

\[ \{1,4\} \]

\[ \{1,2\} \]

\[ \{1,3\} \]

\[ \{1,5\} \]

\[ \{1,6\} \]

\[ \{2,3\} \]

\[ \{2,4\} \]

\[ \{2,5\} \]

\[ \{2,6\} \]

\[ \{3,4\} \]

\[ \{3,5\} \]

\[ \{3,6\} \]

\[ \{4,5\} \]

\[ \{4,6\} \]

\[ \{5,6\} \]

\[ K(6, 2) \]
Example

$f$

$K(6, 2)$
Example

\[ f(\{1,2\}) = 1 \]
\[ f(\{1,3\}) = 0 \]
\[ f(\{1,4\}) = 0 \]
\[ f(\{1,5\}) = 0 \]
\[ f(\{1,6\}) = 0 \]
\[ f(\{2,3\}) = 0 \]
\[ f(\{2,4\}) = 0 \]
\[ f(\{2,5\}) = 0 \]
\[ f(\{2,6\}) = 0 \]
\[ f(\{3,4\}) = 0 \]
\[ f(\{3,5\}) = 0 \]
\[ f(\{3,6\}) = 0 \]
\[ f(\{4,5\}) = 0 \]
\[ f(\{4,6\}) = 0 \]
\[ f(\{5,6\}) = 0 \]

\[ K(6, 2) \]
Cycle factors

\[ K(6, 2) \]
Analysis of cycles

$K(15, 1)$
Analysis of cycles

$K(15, 1)$

$f$

$K(15, 1)$
Analysis of cycles

\[ K(15, 1) \]

- two matched bits form a glider
Analysis of cycles

- two matched bits form a glider
- glider moves by one unit per step
Analysis of cycles

\( K(15, 2) \)

- four matched bits form a **glider**
- glider **moves** by two units per step
Gliders

$K(15, 1)$

$K(15, 2)$

- **glider** = set of matched 1s and 0s
Gliders

\[ K(15, 1) \]

\[ K(15, 2) \]

- **glider** = set of matched 1s and 0s
- **speed** \((\nu)\) = numbers of 1s = number of 0s
Motion of gliders

$K(15, 3)$

$v = 1$  $v = 2$
Motion of gliders

$K(15, 3)$

$v = 1$  
$v = 2$

$t$

$v = 1$  
$v = 2$
Motion of gliders

$K(15, 3)$

\[ \begin{align*}
\text{\textbf{position on zero}} &: s(t) = v \cdot t + s(0) \\
\text{\textbf{speed}} &: \text{number of times } f \text{ is applied}
\end{align*} \]
Motion of gliders

\[ K(15, 3) \]

\[ s(t) = v \cdot t + s(0) \]

- position after time \( t \): \[ s(t) = v \cdot t + s(0) \]
- motion can be either uniform or non-uniform

\( v = 1 \)

\( v = 2 \)
Gliders partitioning

$K(18, 7)$

classical matching
Gliders partitioning

\(K(18, 7)\)

- Assigning matched bits to gliders (coloring boxes) is complicated!
Gliders partitioning

$K(18, 7)$

• Assigning matched bits to gliders (coloring boxes) is complicated!

• recursion assigns the matched bits to gliders
Gliders partitioning

$K(18, 7)$

- Assigning matched bits to gliders (coloring boxes) is complicated!

- Recursion assigns the matched bits to gliders

- Speed set of gliders is a **cycle invariant**
Gluing cycles

\[ C_1 \xrightarrow{f} f(x) \xleftarrow{f} f(y) \]

\[ C_2 \xrightarrow{f} f(y) \xleftarrow{f} y \]
Gluing cycles

- pair \((x, y)\) such that \(xf(x) yf(y)\) is a 4-cycle
Gluing cycles

- pair \((x, y)\) such that \(xf(x) yf(y)\) is a 4-cycle
Gluing cycles

- pair \((x, y)\) such that \(x f(x) y f(y)\) is a 4-cycle
  - 4-cycle is there since \(n \geq 2k + 3\)
Gluing cycles
Gluing cycles

- gluing cycles must be edge-disjoint
Gluing cycles

- find \((x, y)\) when the speed set of gliders increases lexicographically
Gluing cycles

- find \((x, y)\) when the speed set of gliders increases lexicographically
Gluing cycles

- find \((x, y)\) when the speed set of gliders increases lexicographically

\[
x + f(x) = f(y) + y
\]

- ensures connectivity and hence Hamiltonicity
Open questions

• efficient algorithms?
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- efficient algorithms?
- other vertex transitive graphs?
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• efficient algorithms?
• other vertex transitive graphs?
• Hamilton decomposition of Kneser graphs?
Thank you!