Kneser graphs are Hamiltonian

Namrata (Warwick)
joint work with
Arturo Merino (TU Berlin)
Torsten Mütze (Warwick and Charles University, Prague)

Scottish Combinatorics Meeting 2023

Introduction

- Kneser graph $K(n, k)$

  vertices : k-elements subsets of {1, 2,…, n}

  edges : (A, B), where $A \cap B = \emptyset$

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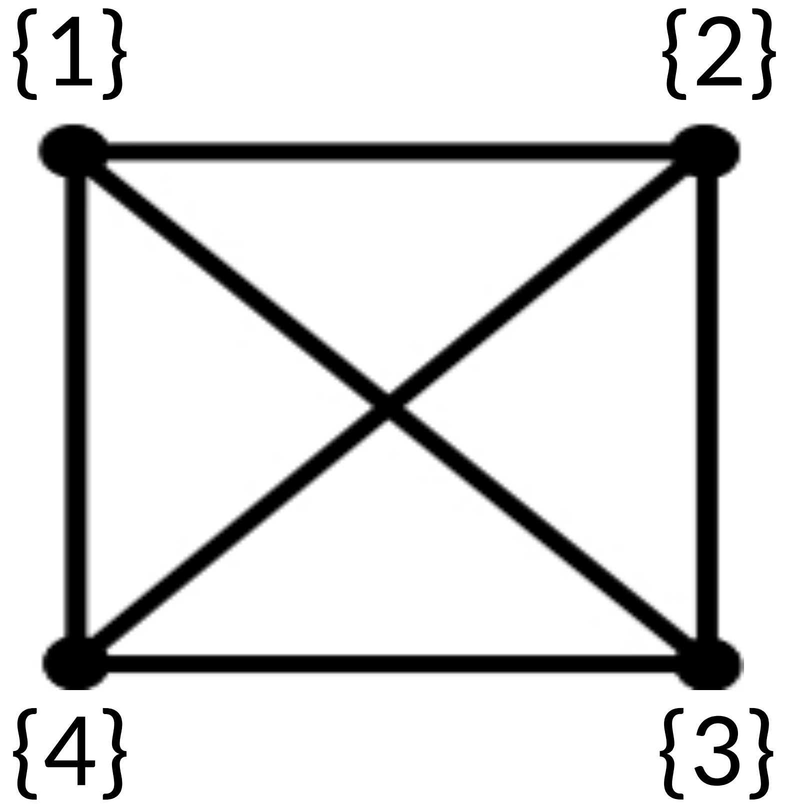

Complete graph $K(4,1)$

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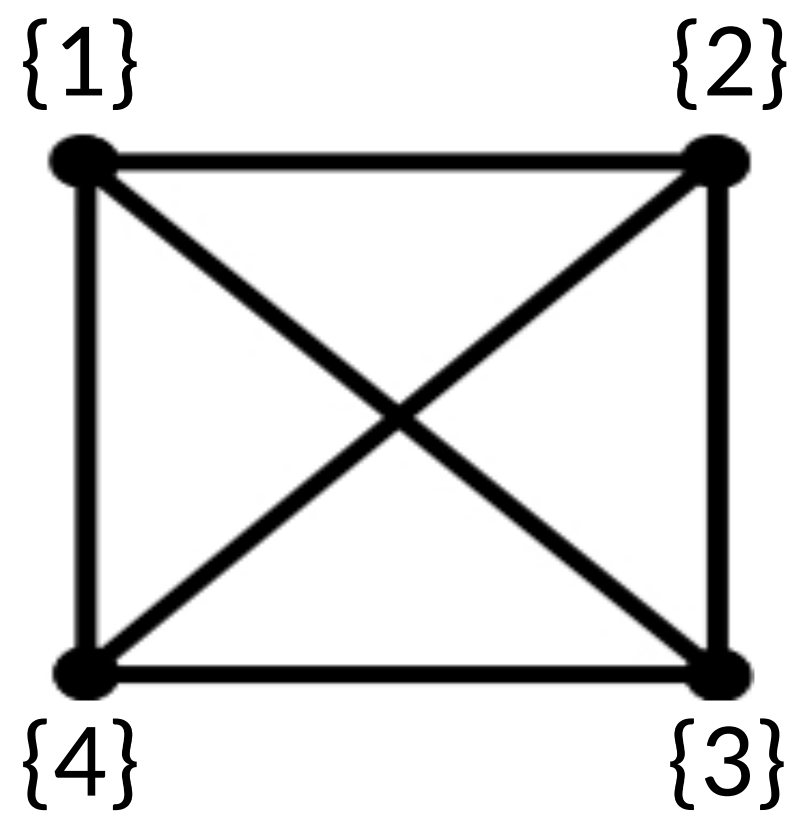

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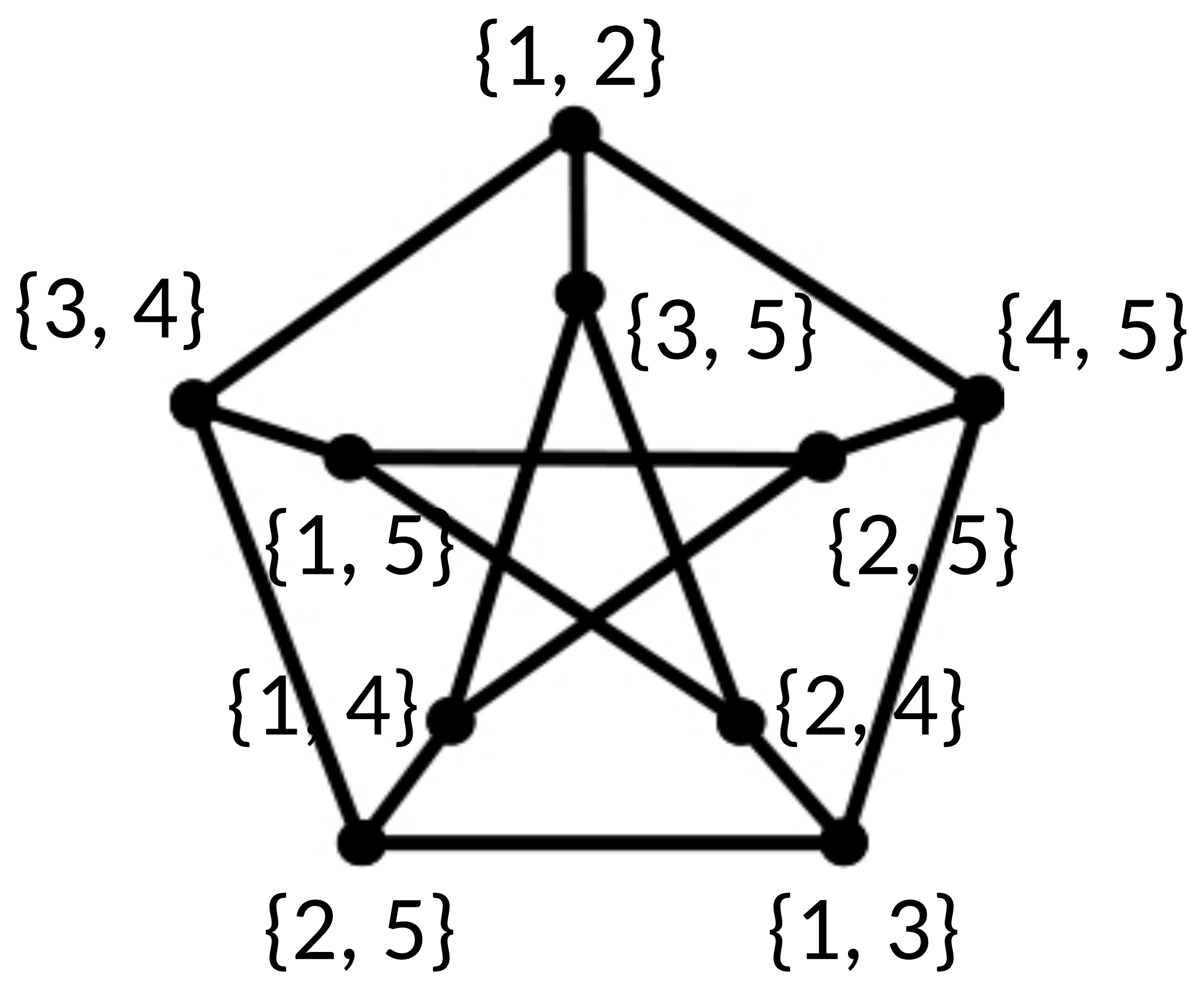

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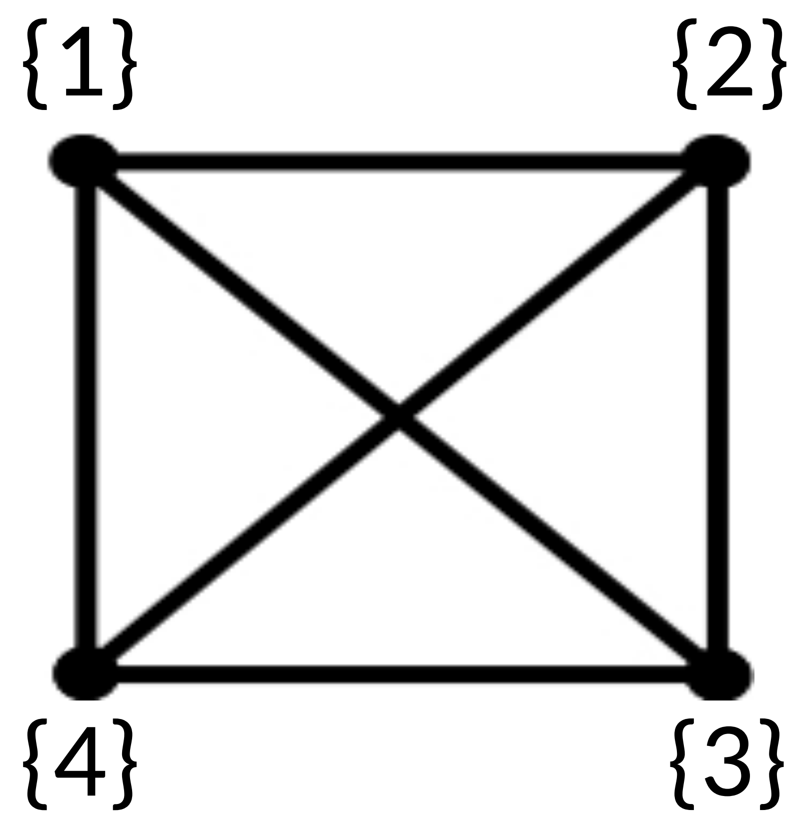
- assumption : $k \geq 1$ and $n \geq 2k + 1$



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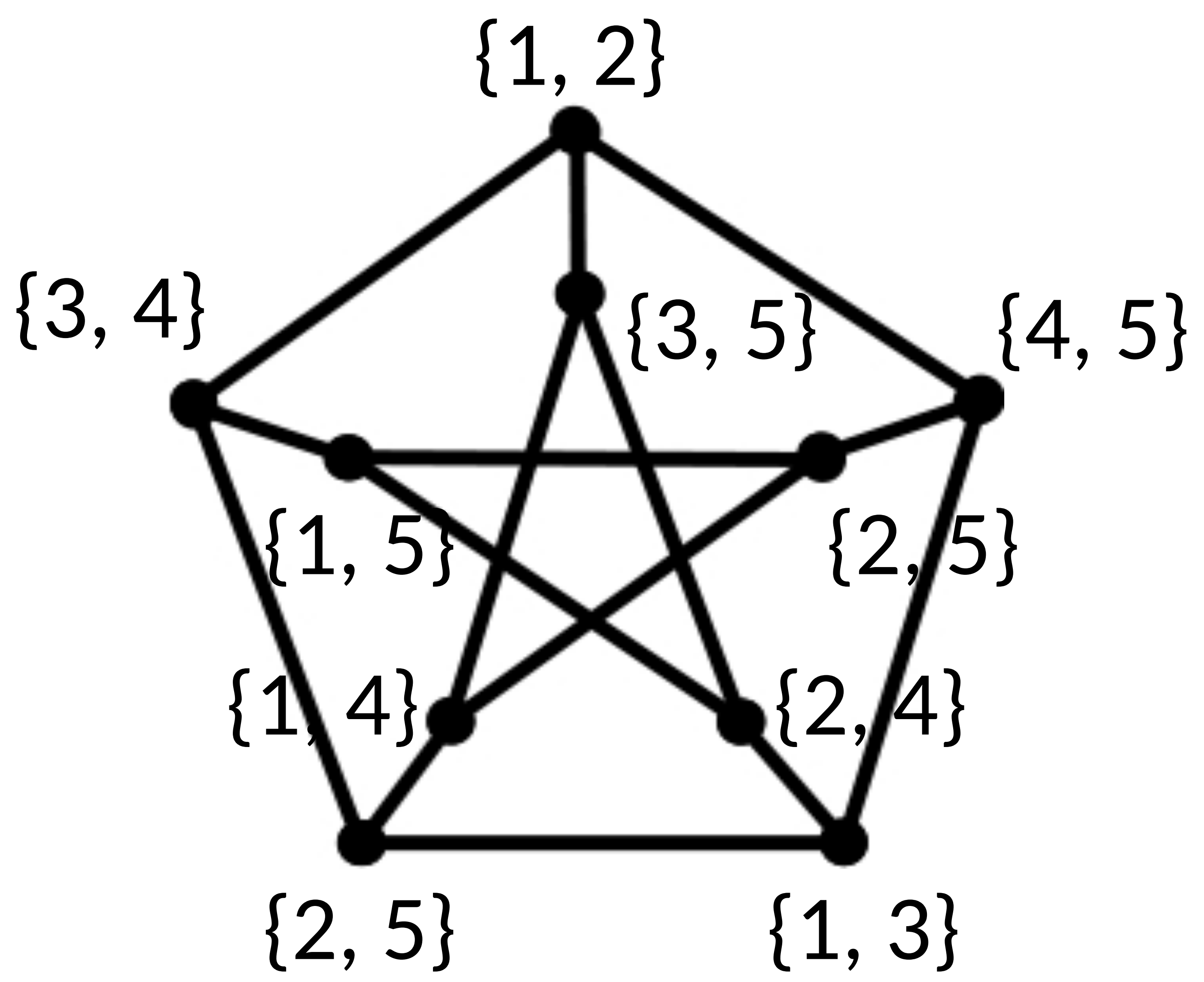

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- special case : Kneser graph, with Petersen graph $K(5,2)$ as an exception

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- [Y. Chen+Füredi 2002]: short proof for $n = ck, c \in \{3,4,\ldots\}$

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- open: $2k + 3 \leq n \leq (1 + o(1))\, 2.62\, k$, where $n \neq 2k + 2^a$

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- settles Hamiltonicity of $K(n,k)$ in full generality

Generalized Johnson graphs

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- settles the Hamiltonicity problem for graphs defined by intersecting set-systems

Proof outline

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- glue cycles together (the assumption $n \geq 2k+3$ is important)

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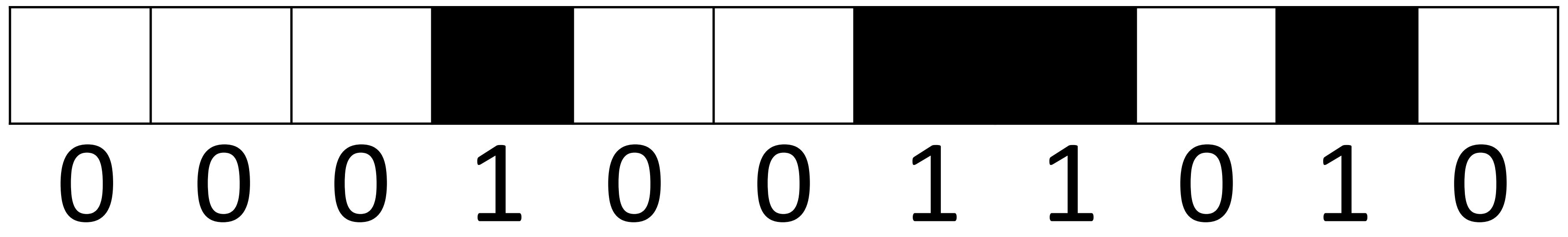
- Example: $n = 11, k = 4,$ x = {4, 7, 8, 10}

x



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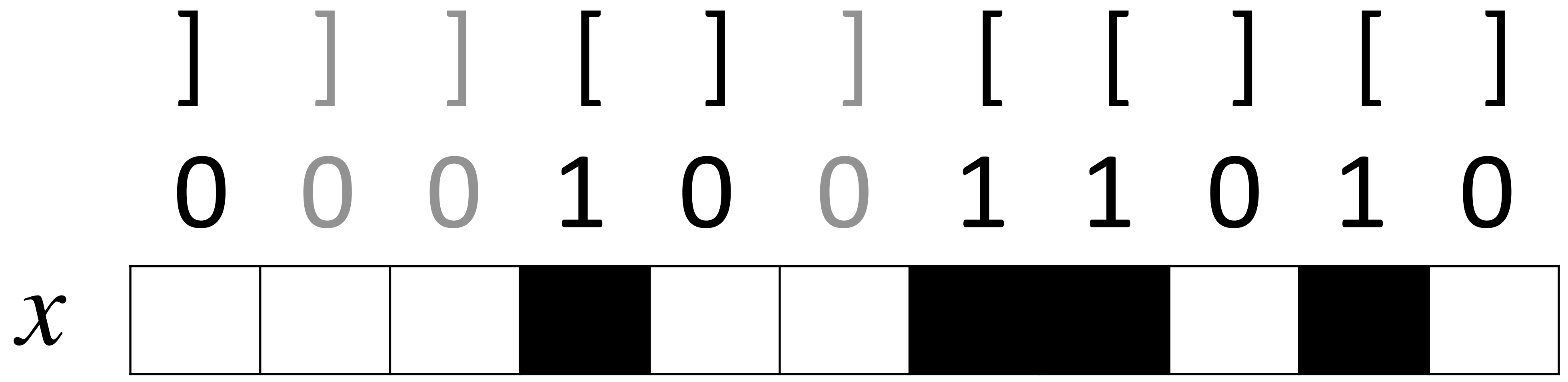

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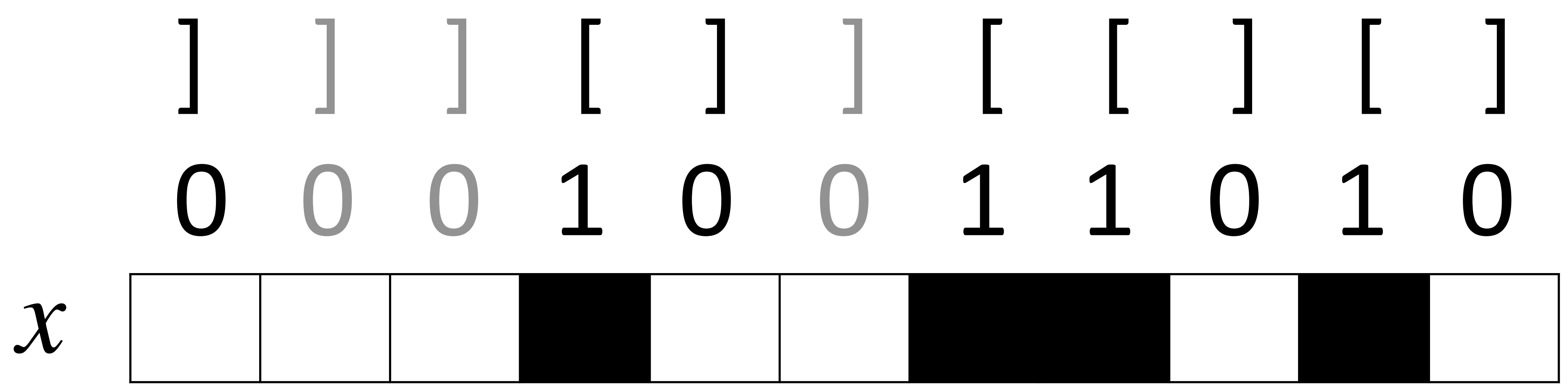

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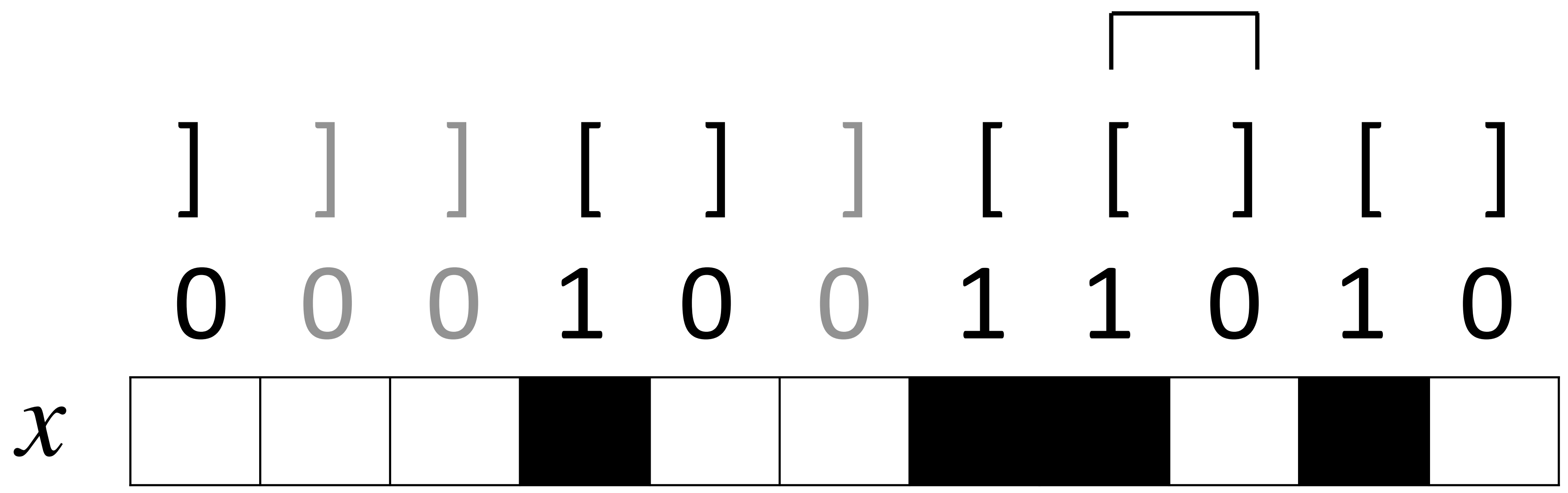

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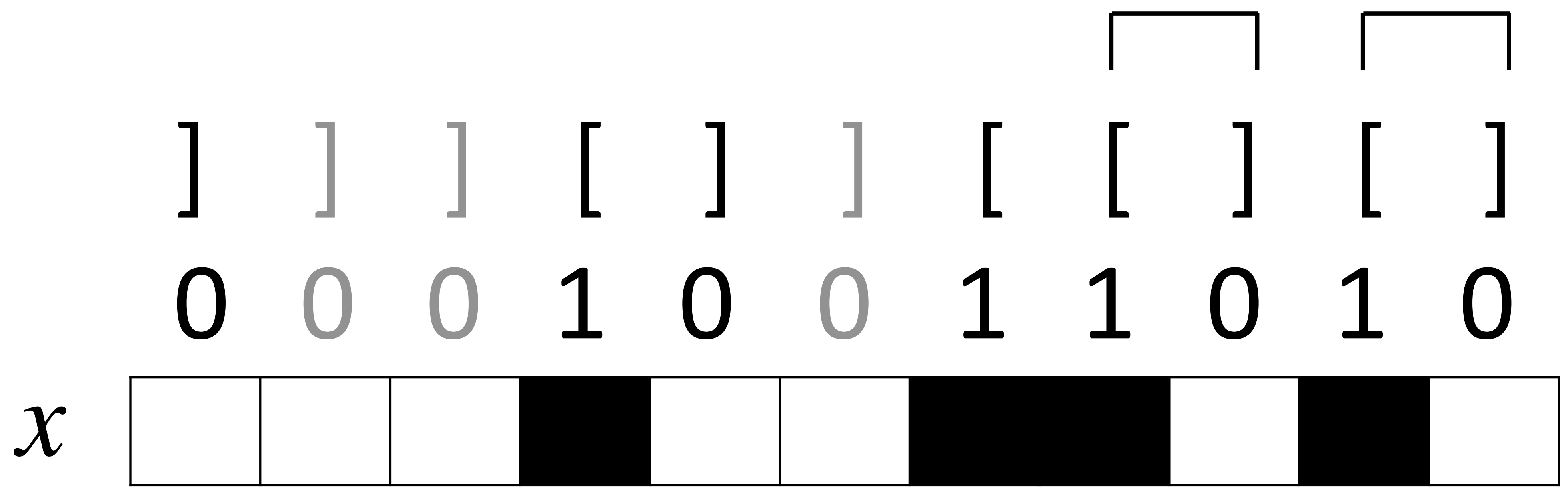

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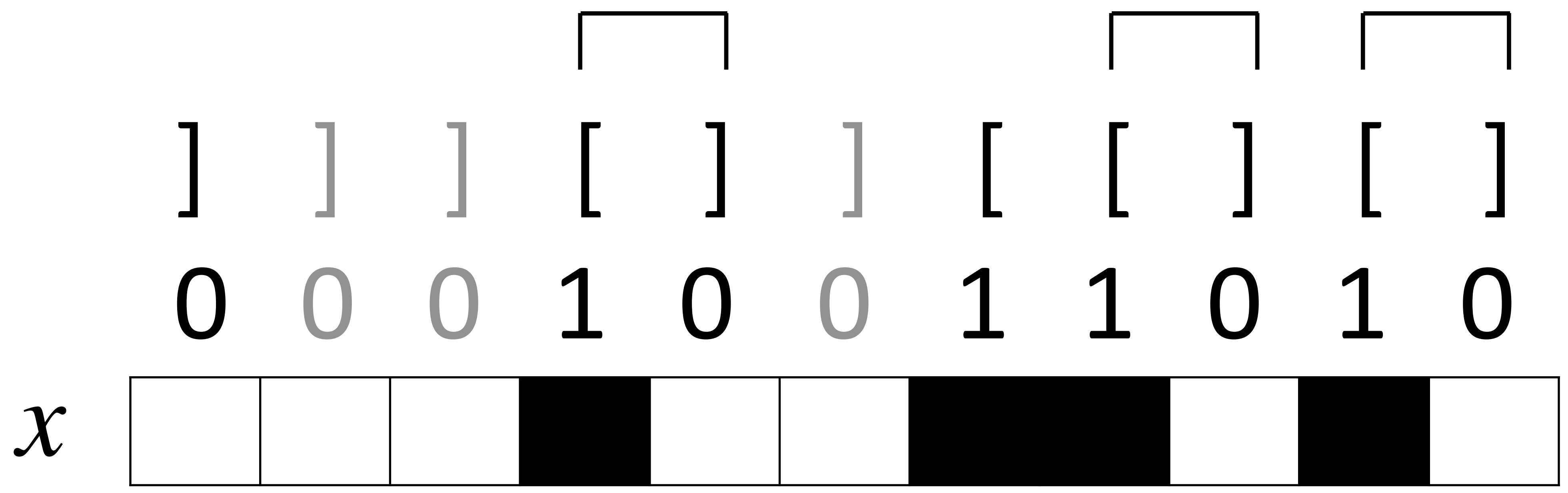

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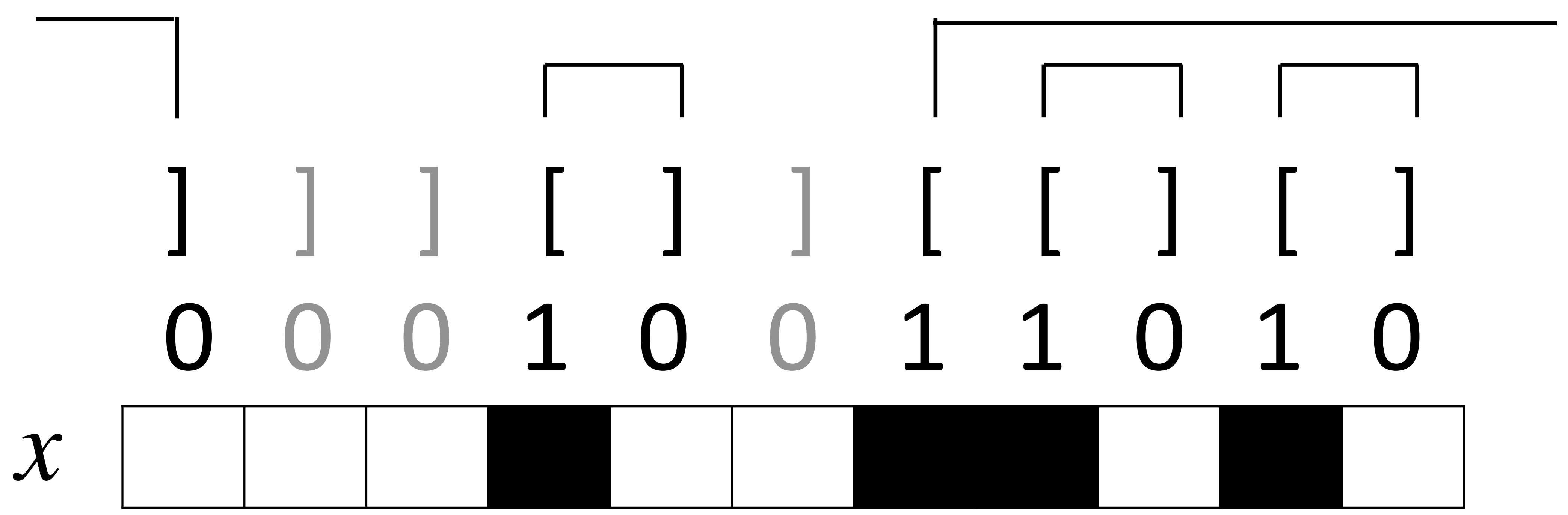

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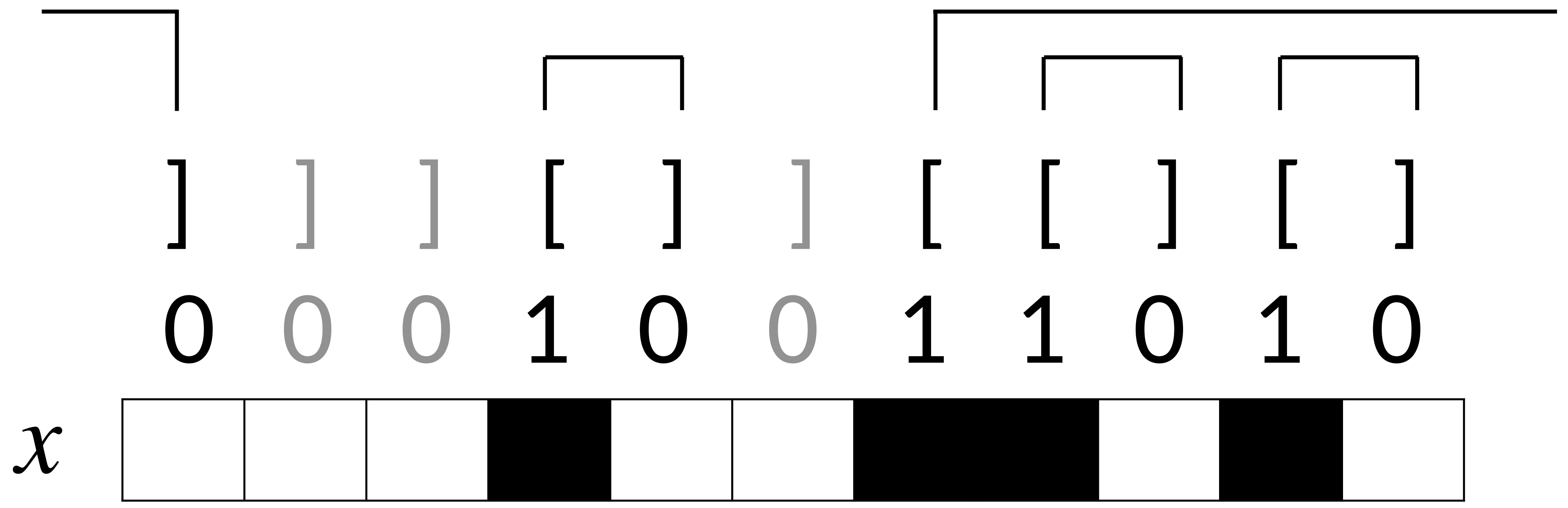

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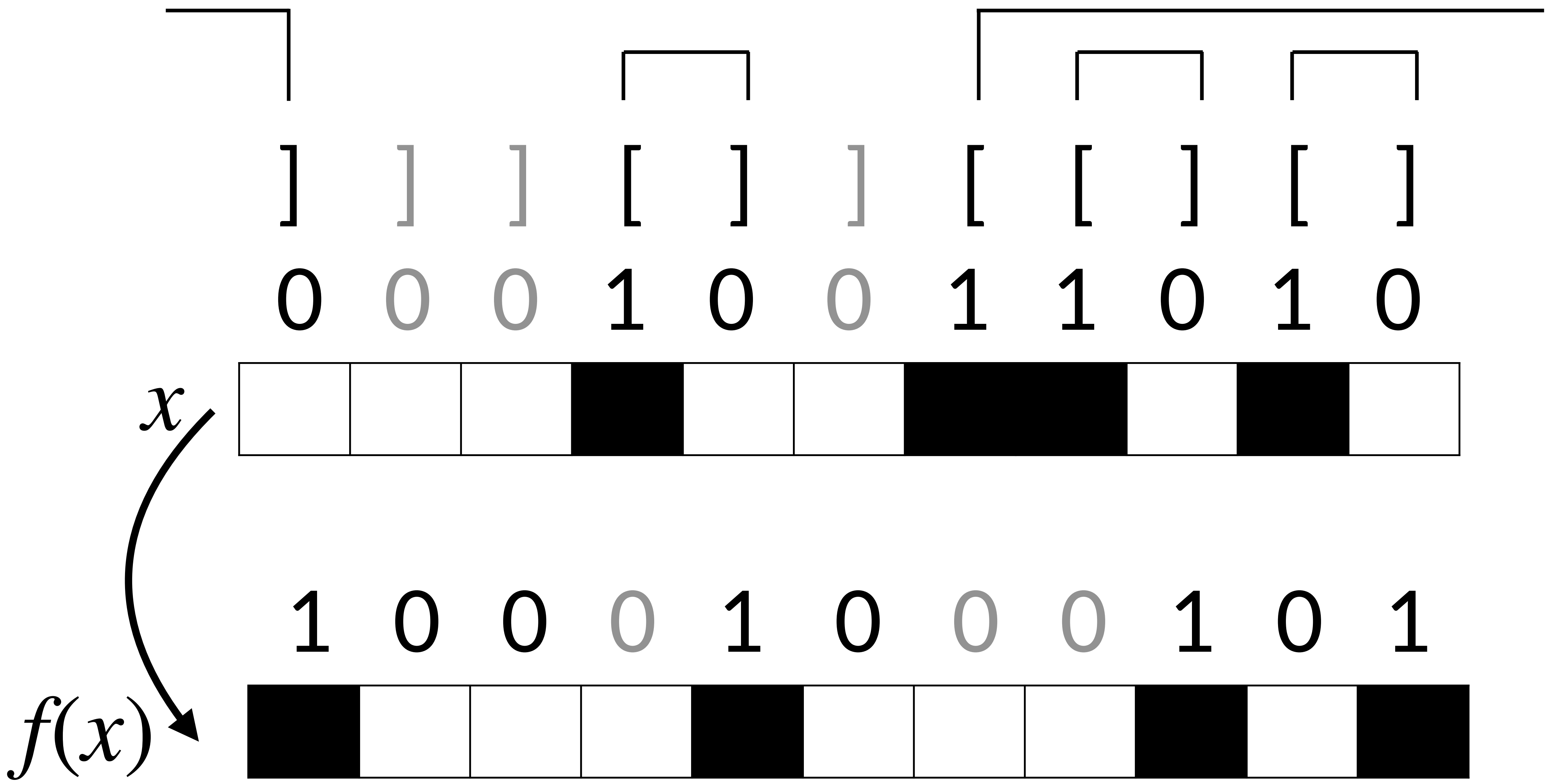

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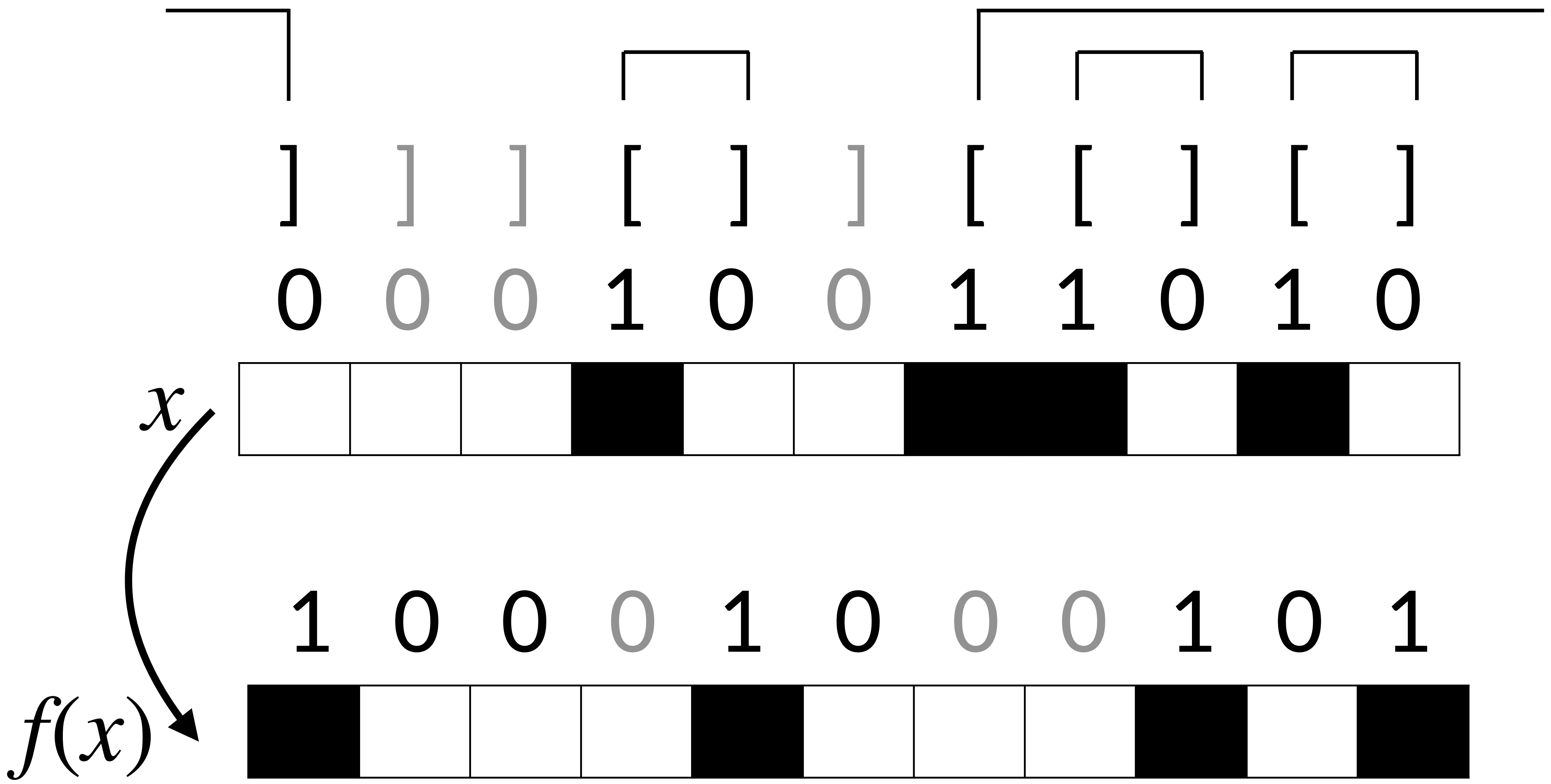

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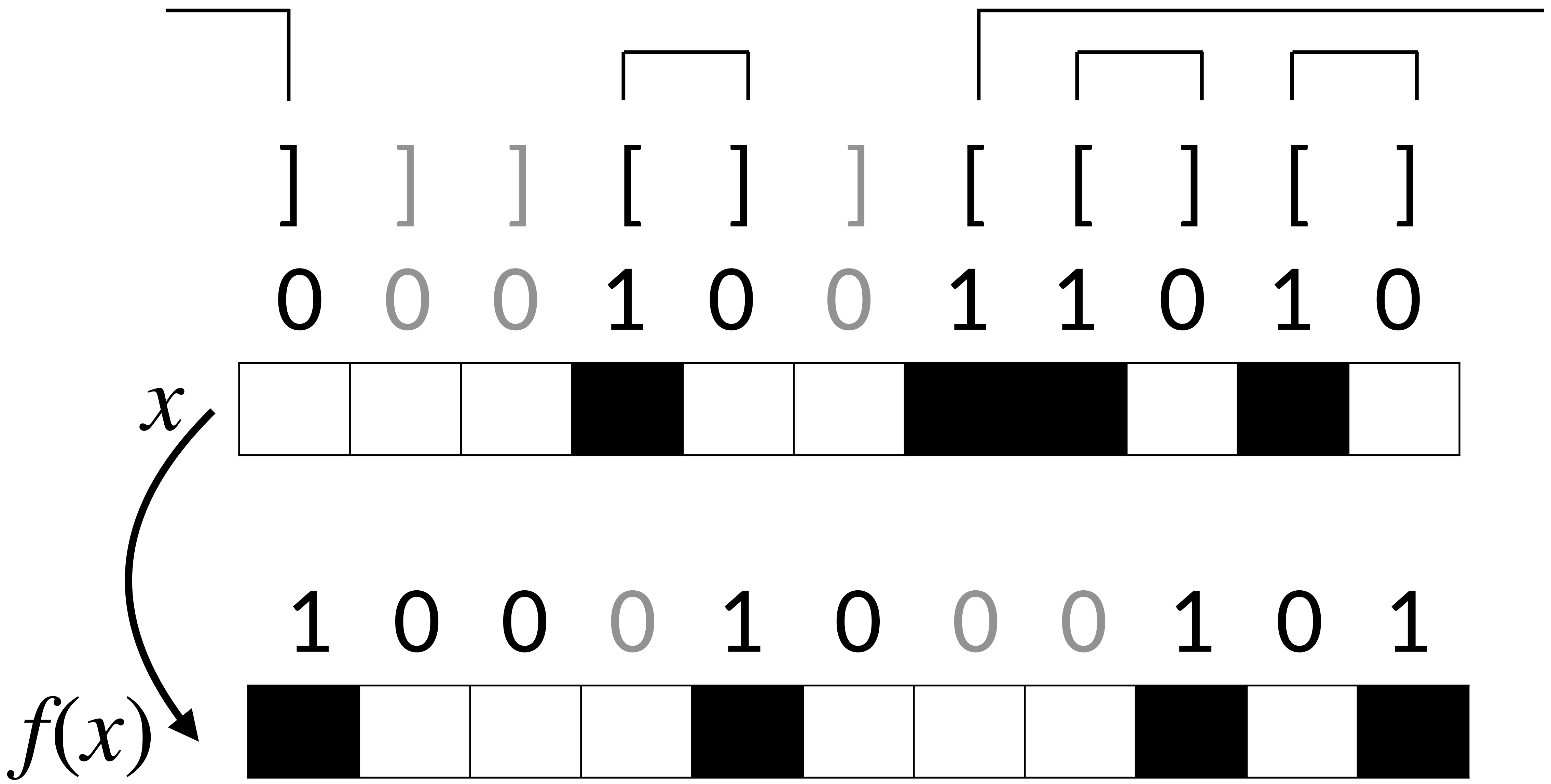

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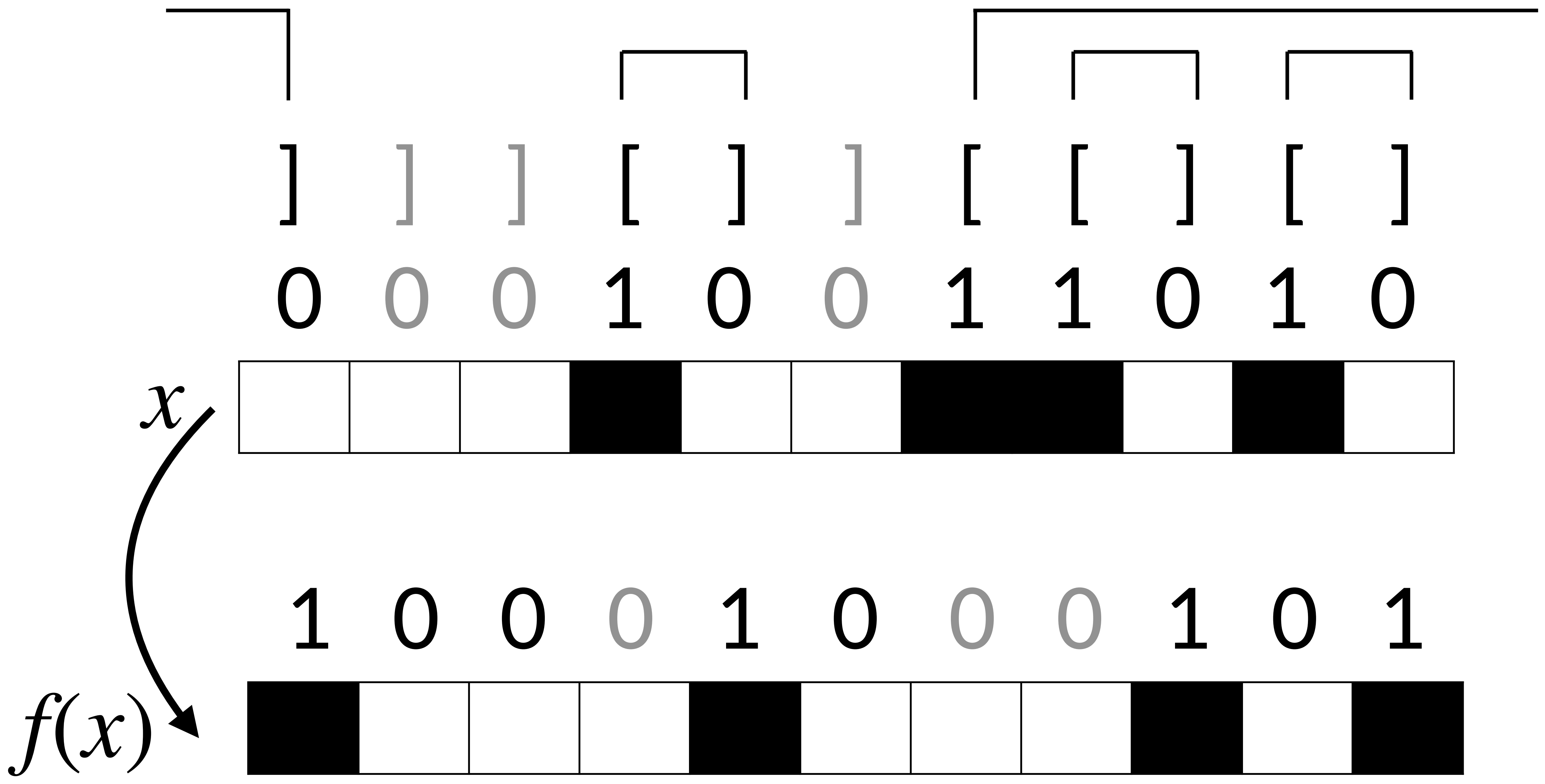

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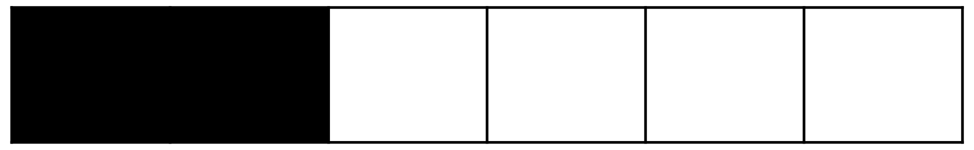
- partitions the vertices of $K(n, k)$ into disjoint cycles

Example



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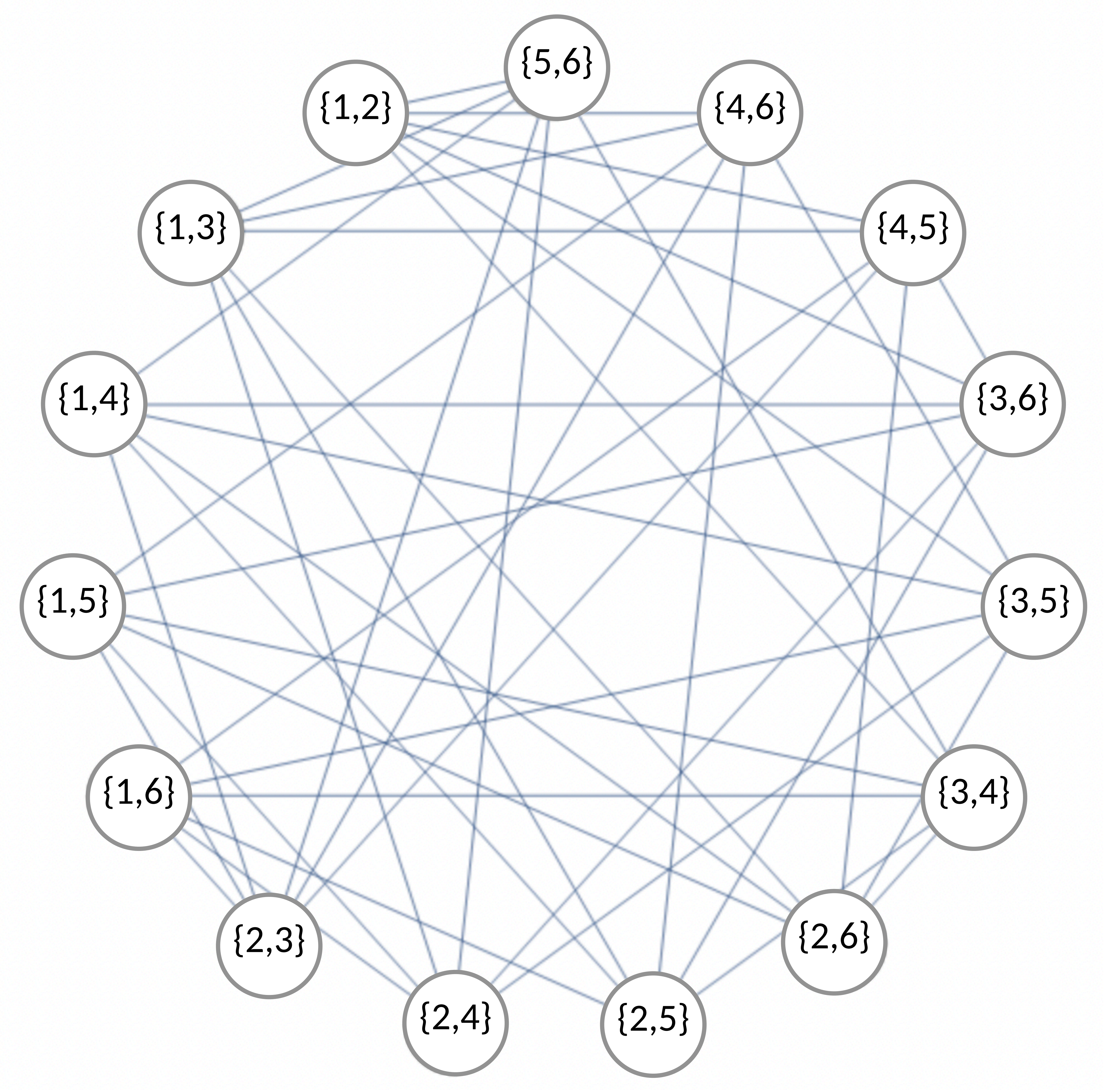

$K(6, 2)$

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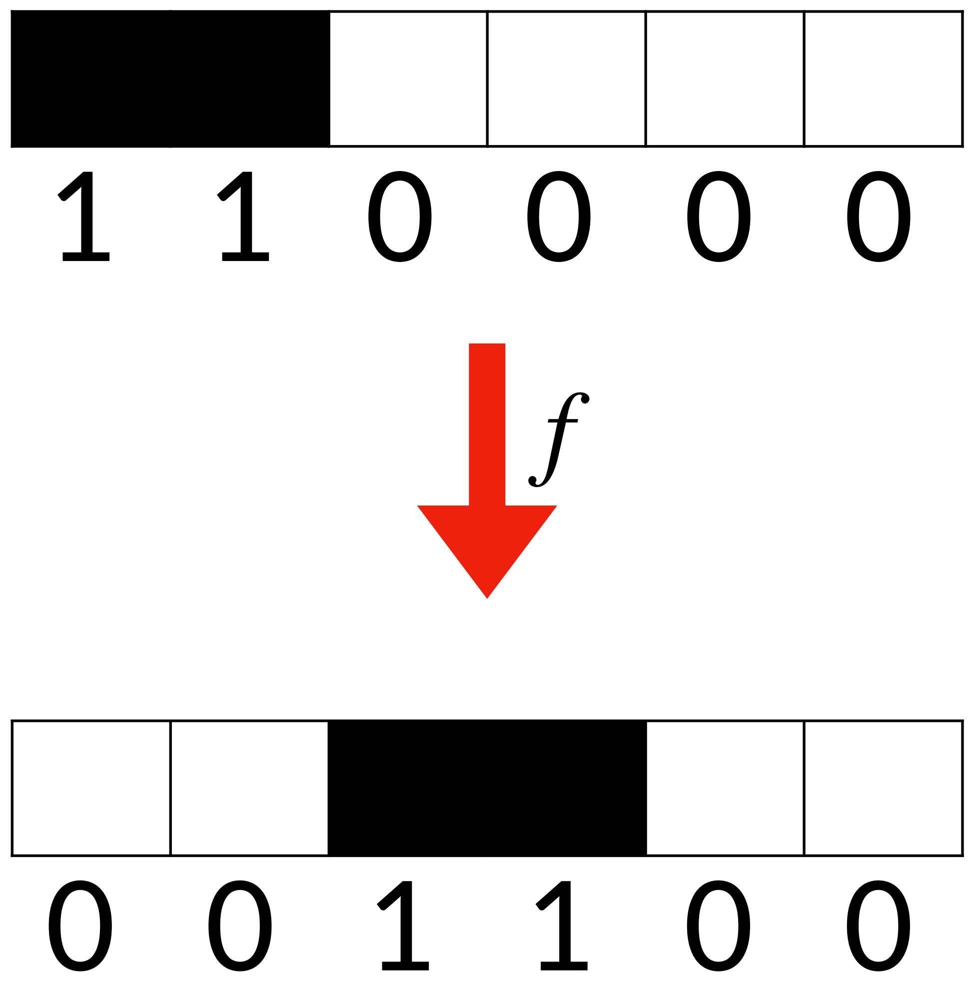

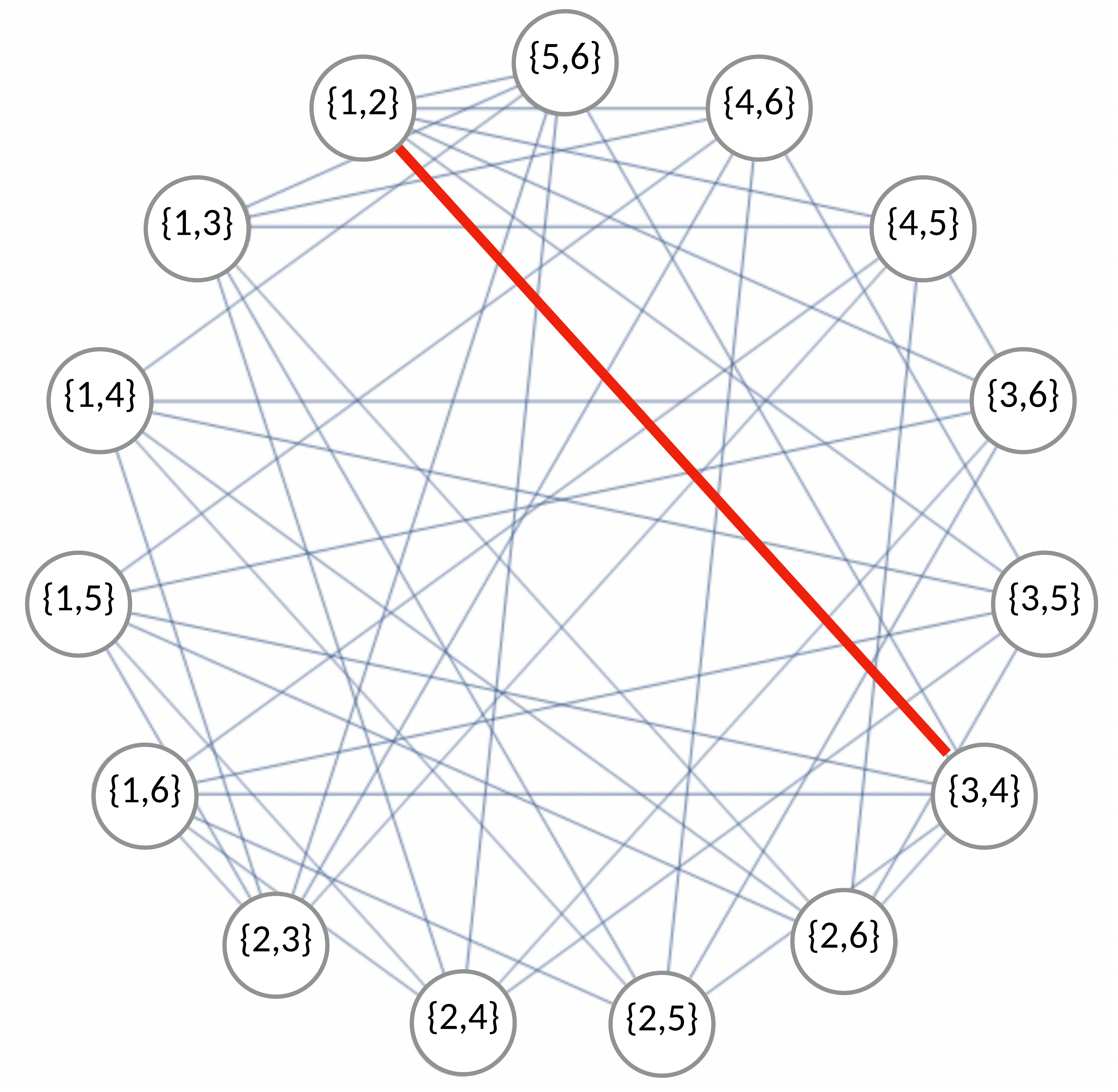

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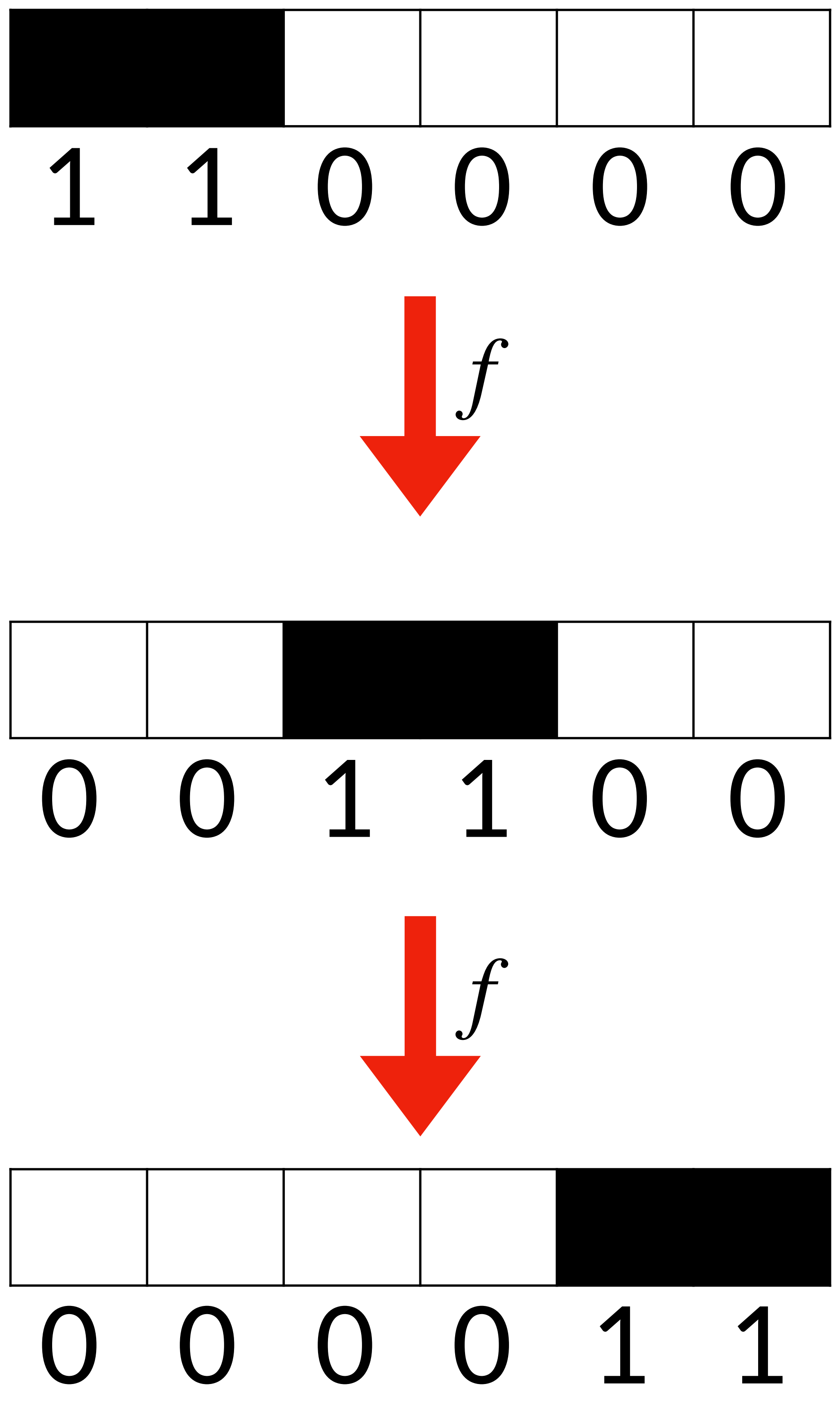
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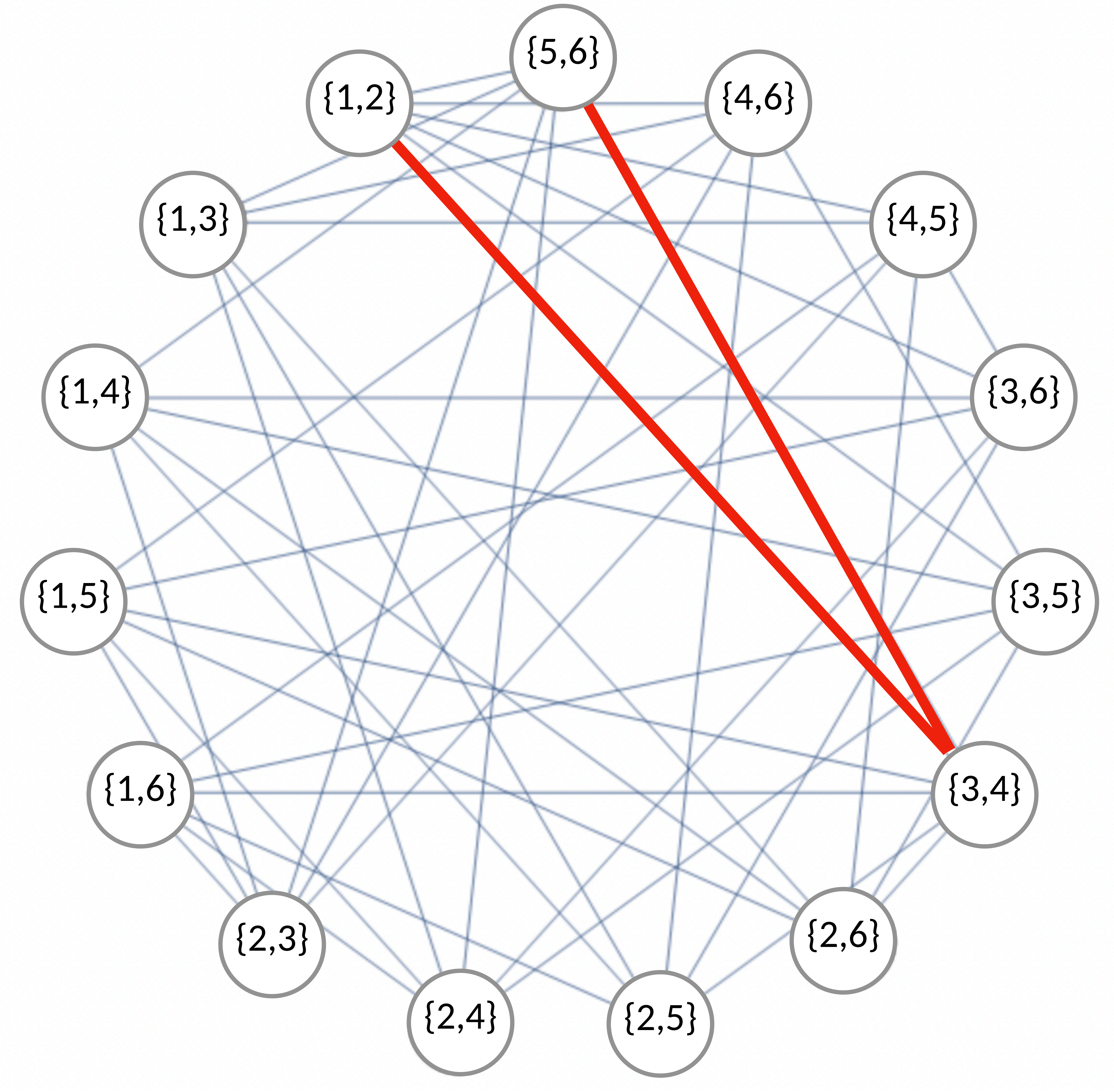
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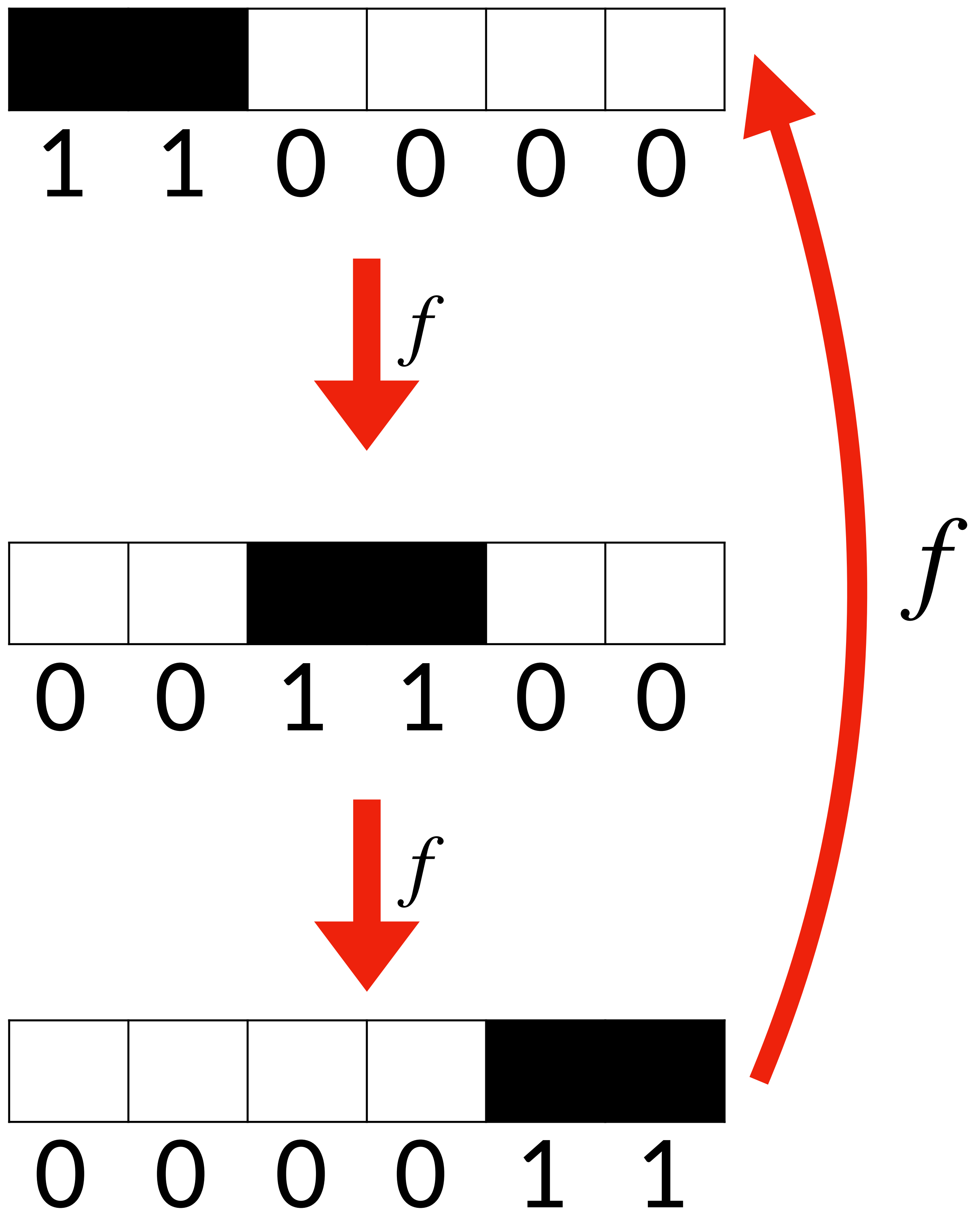
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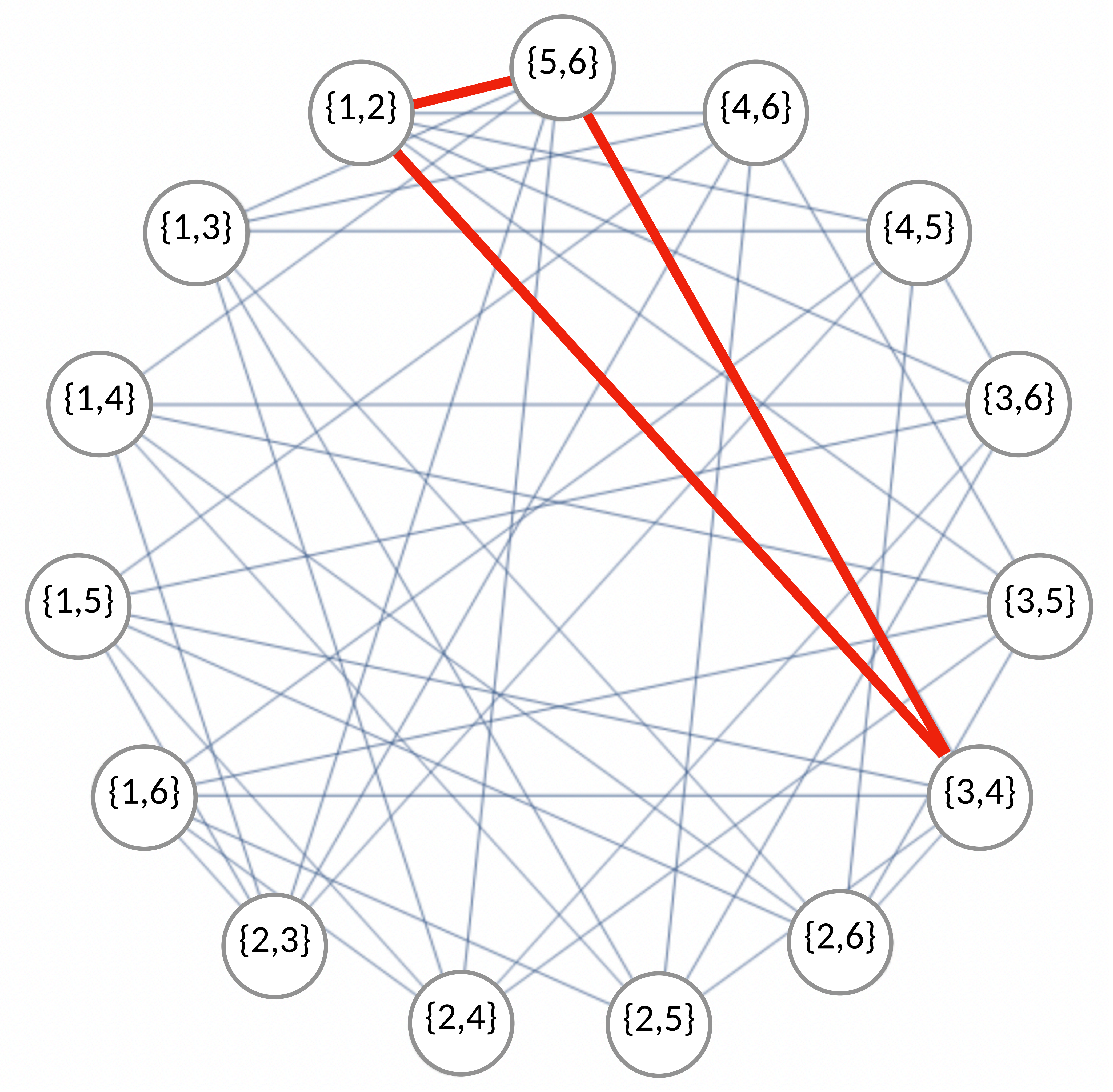
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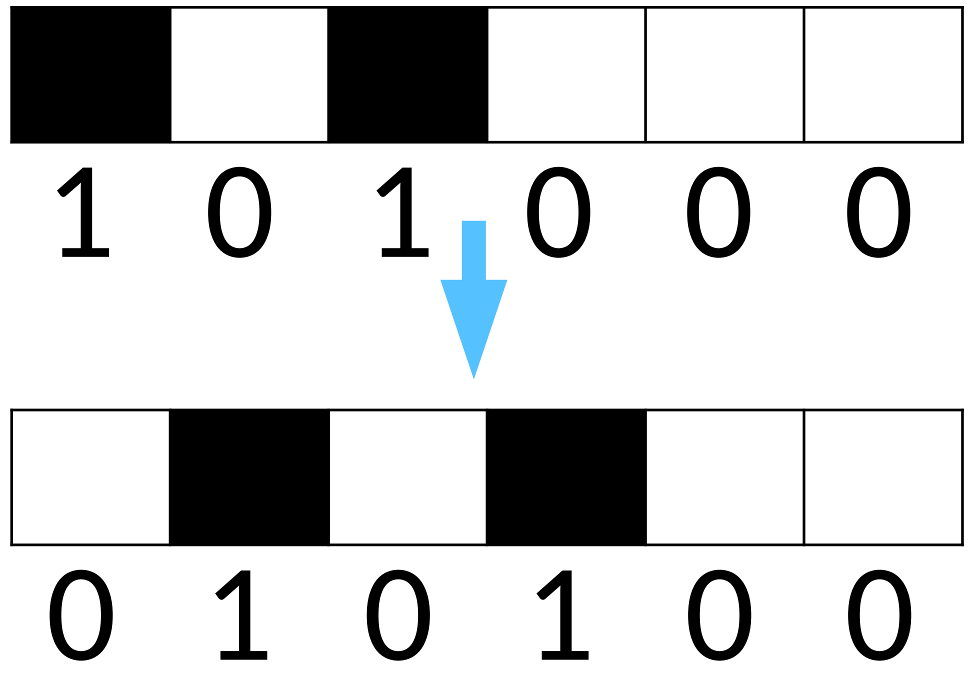

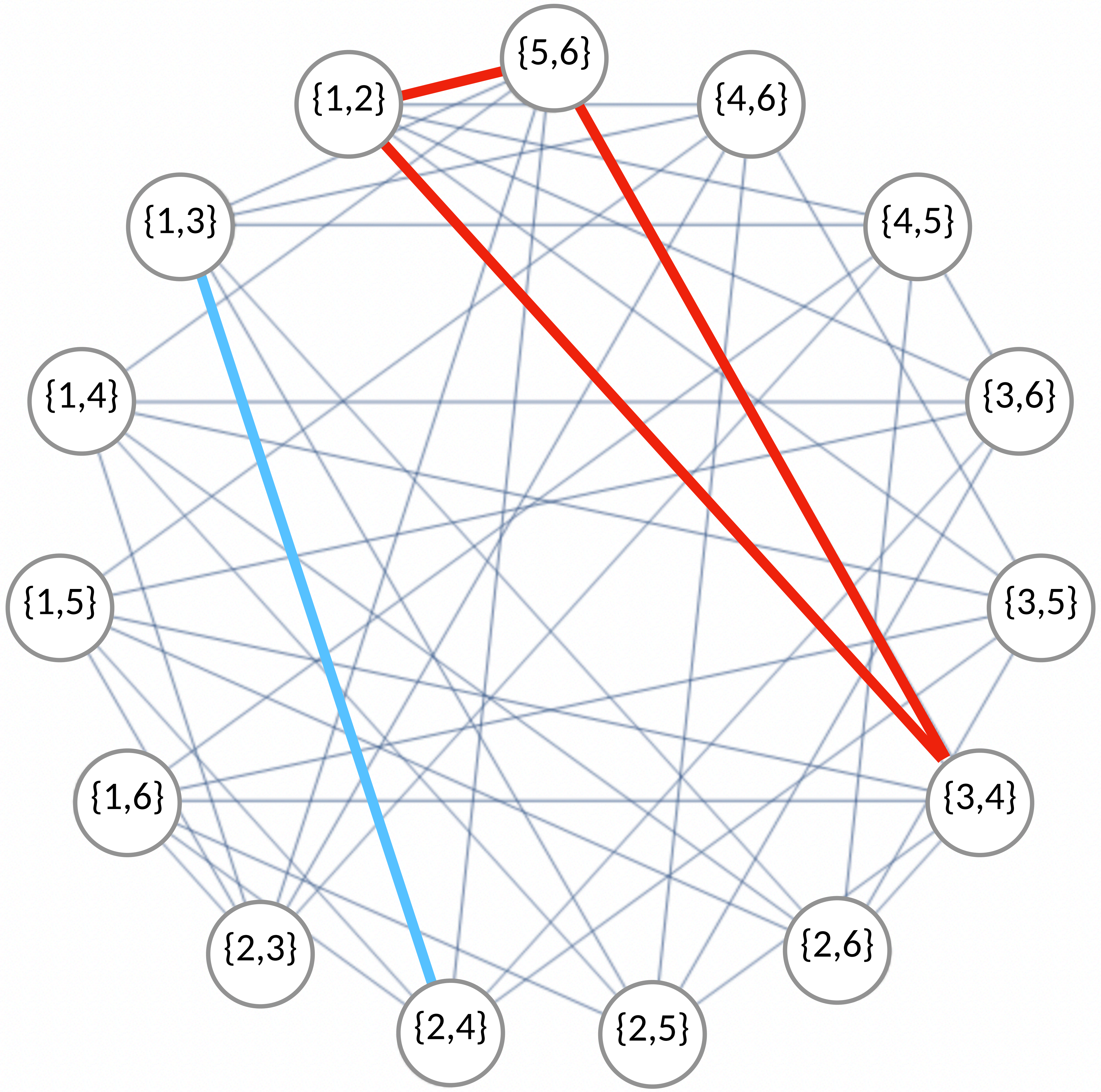

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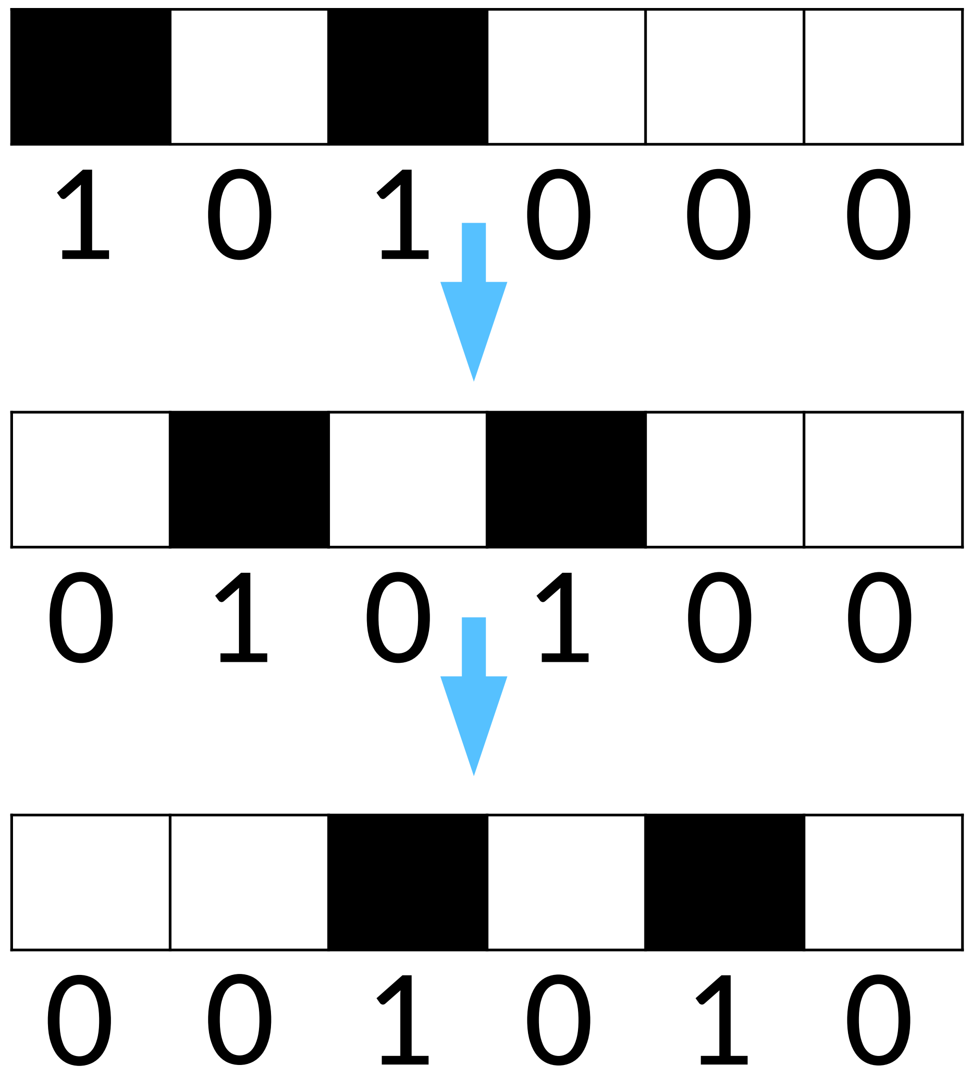

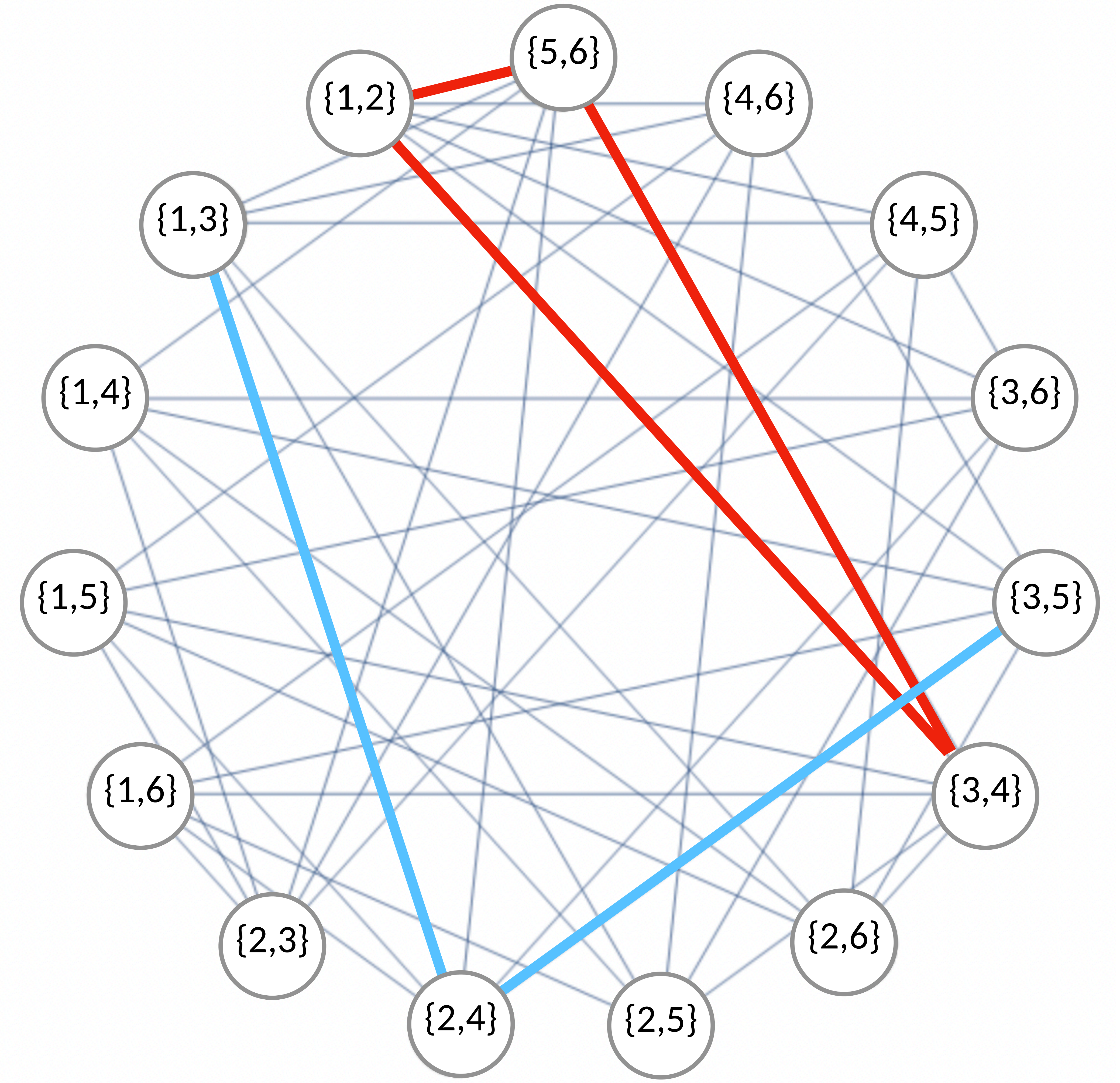

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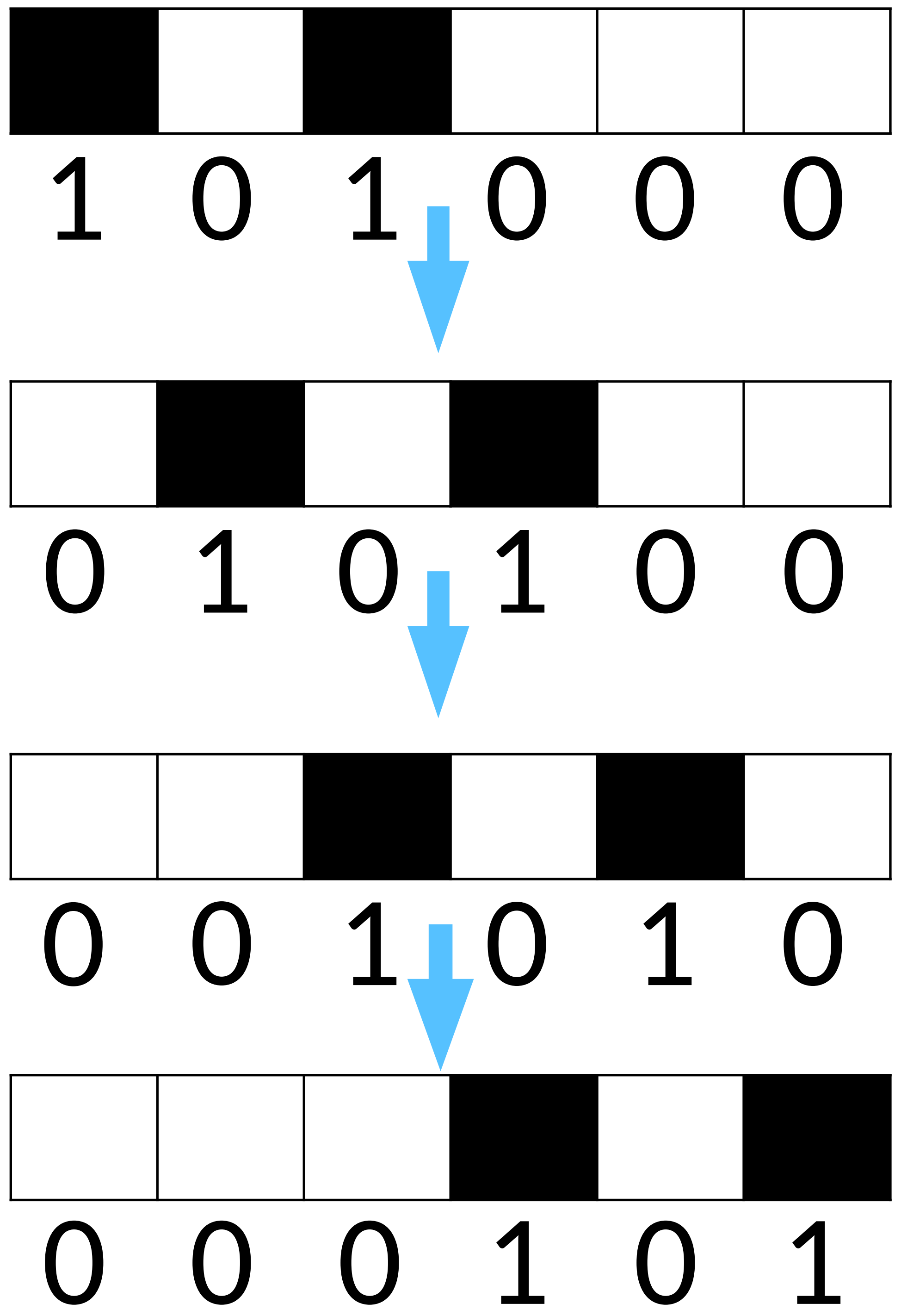
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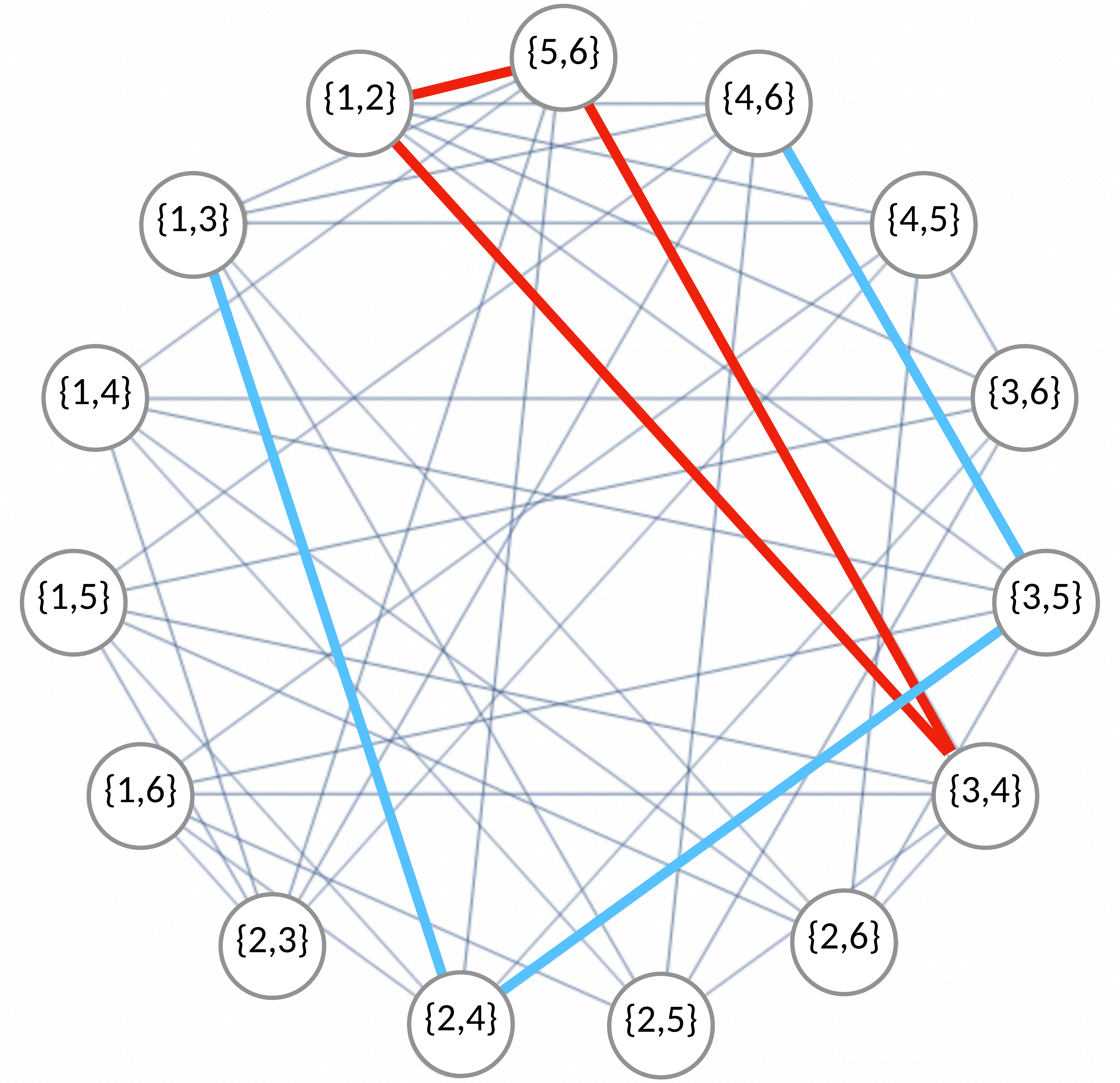
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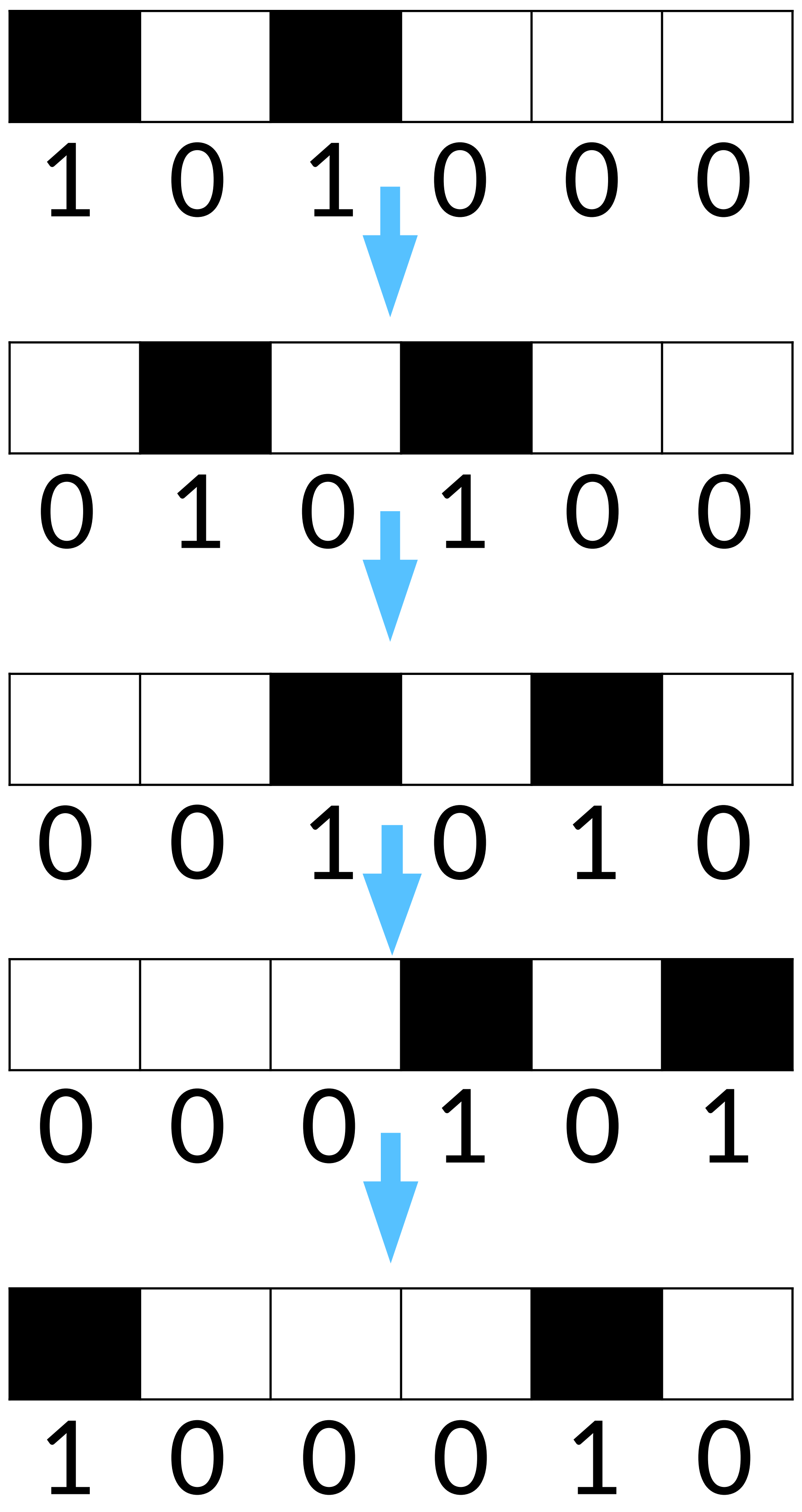
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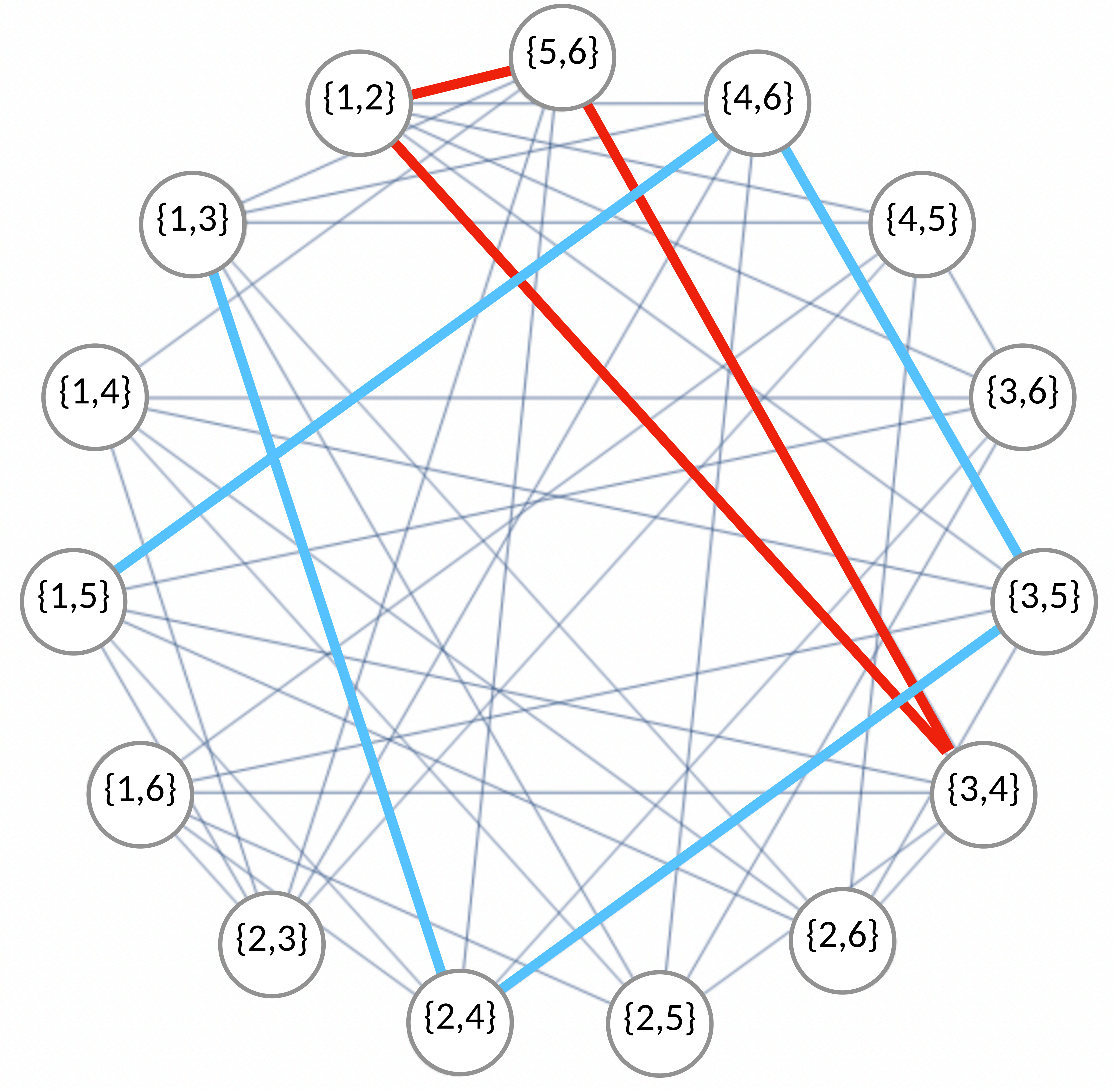
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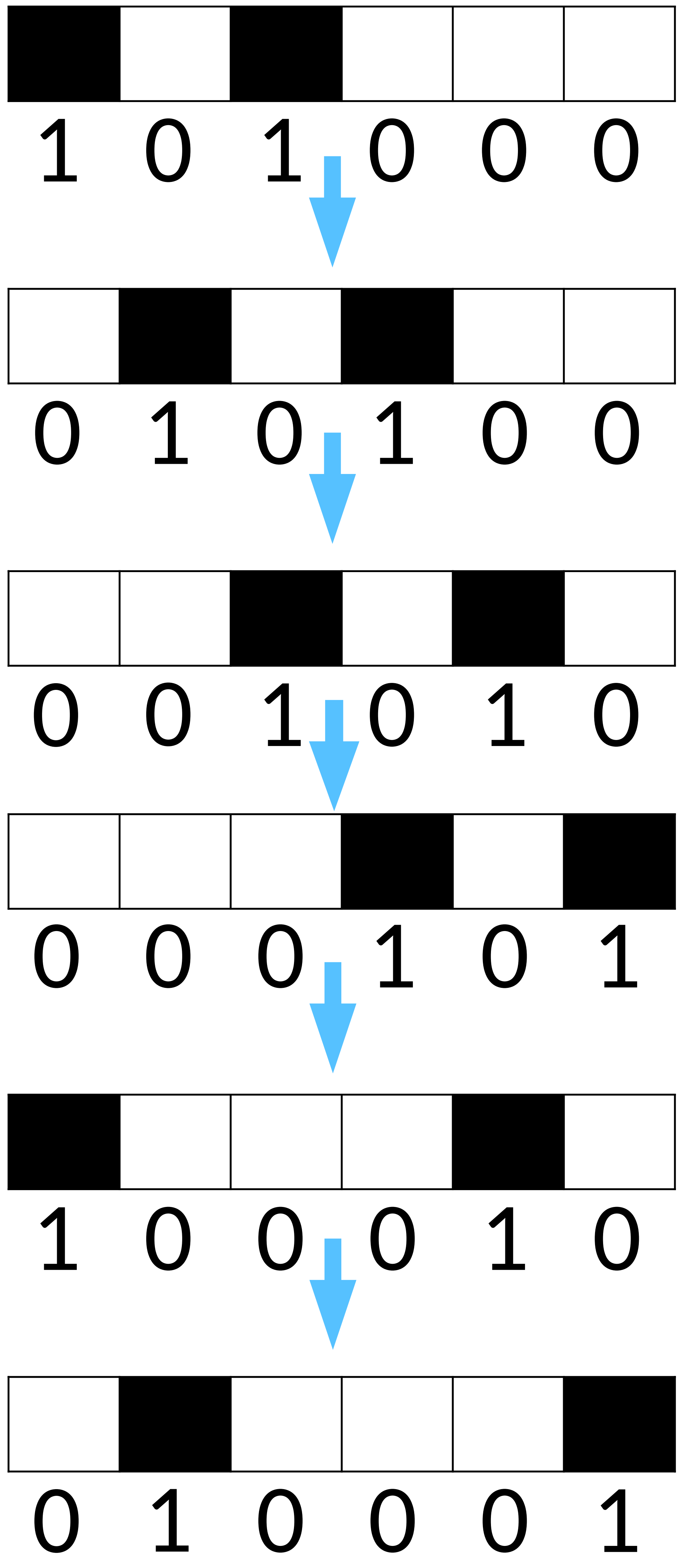
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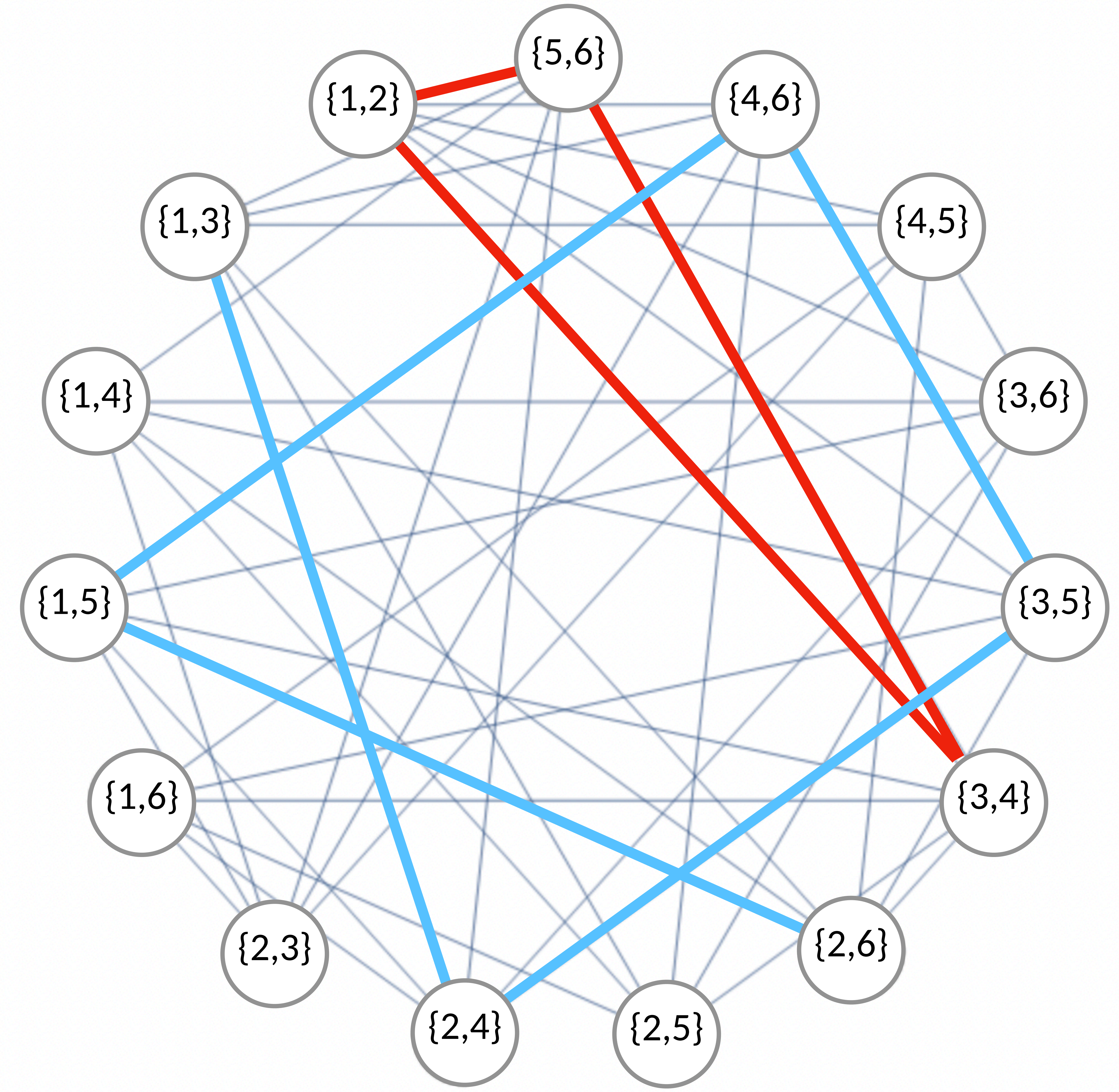
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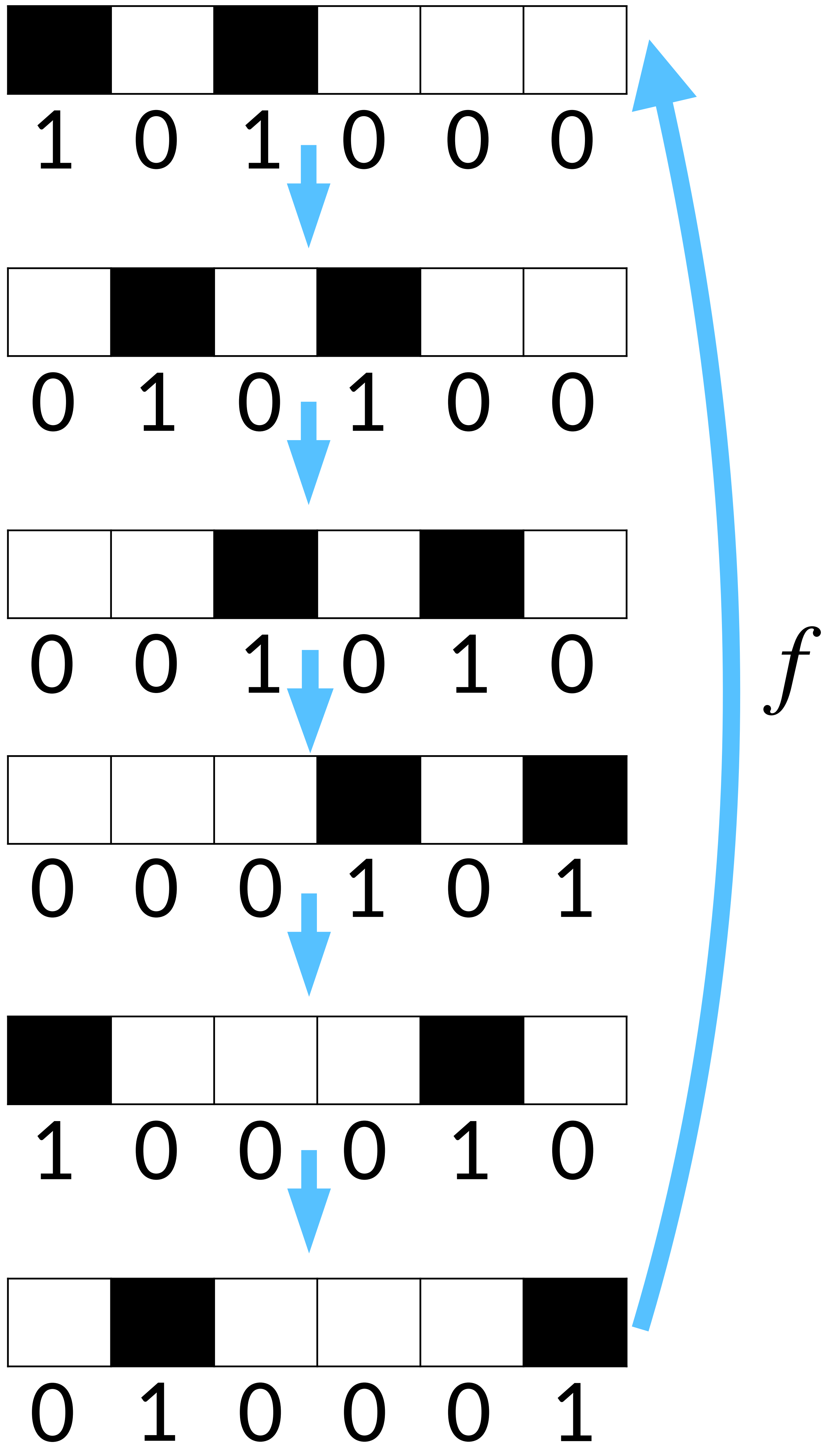

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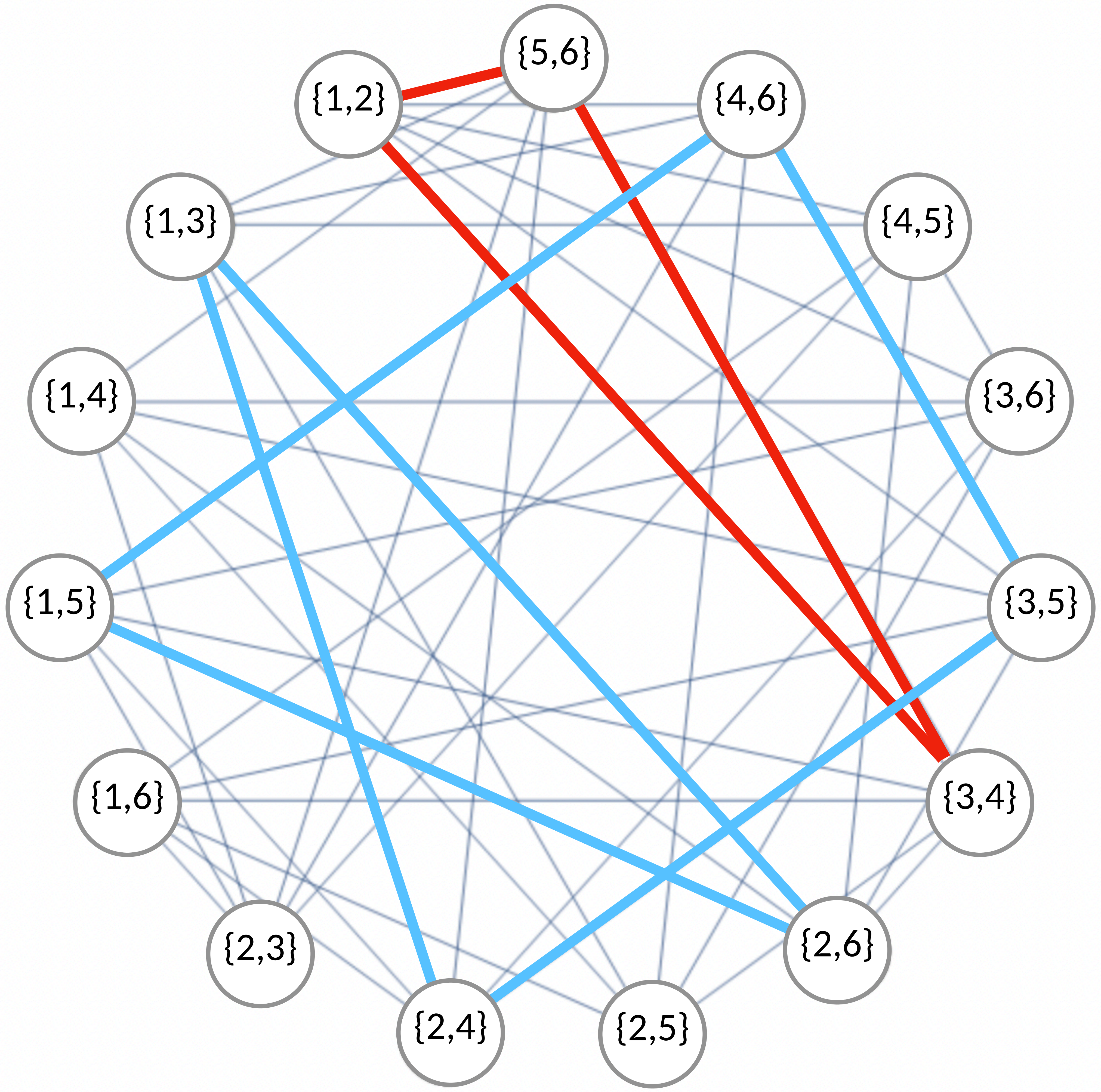

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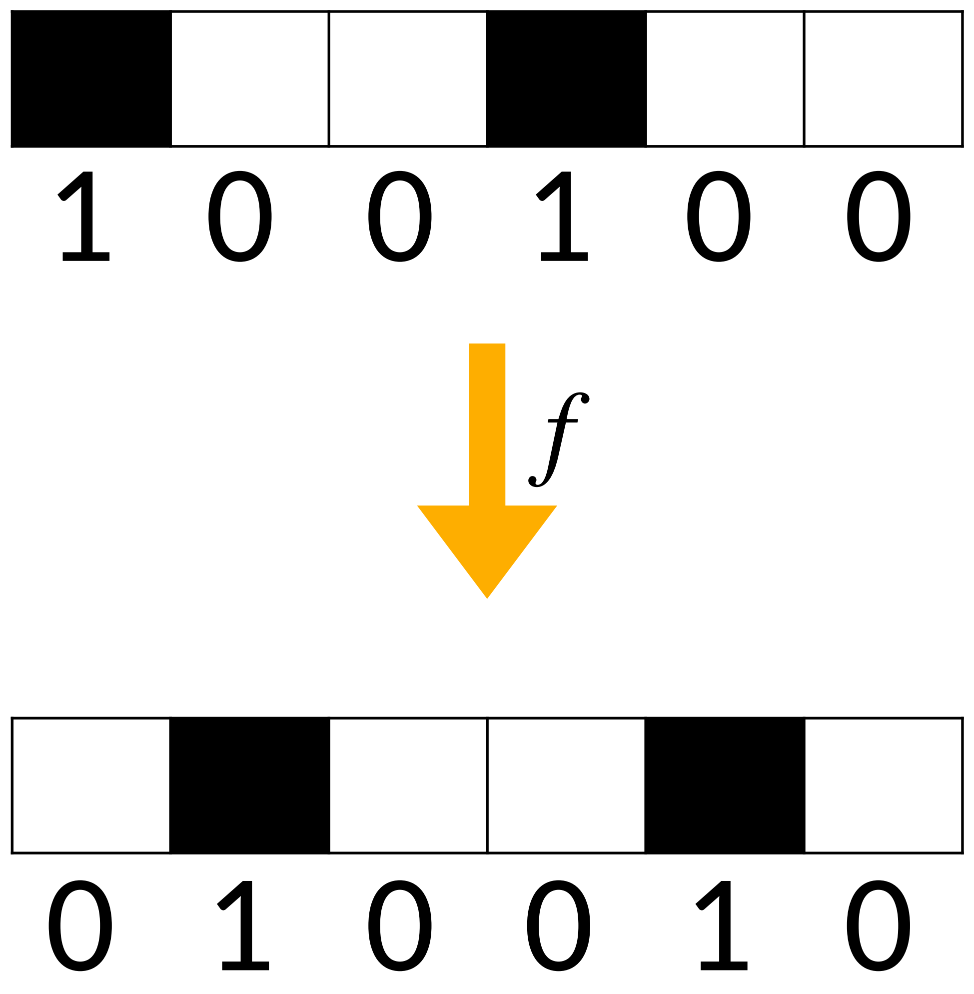
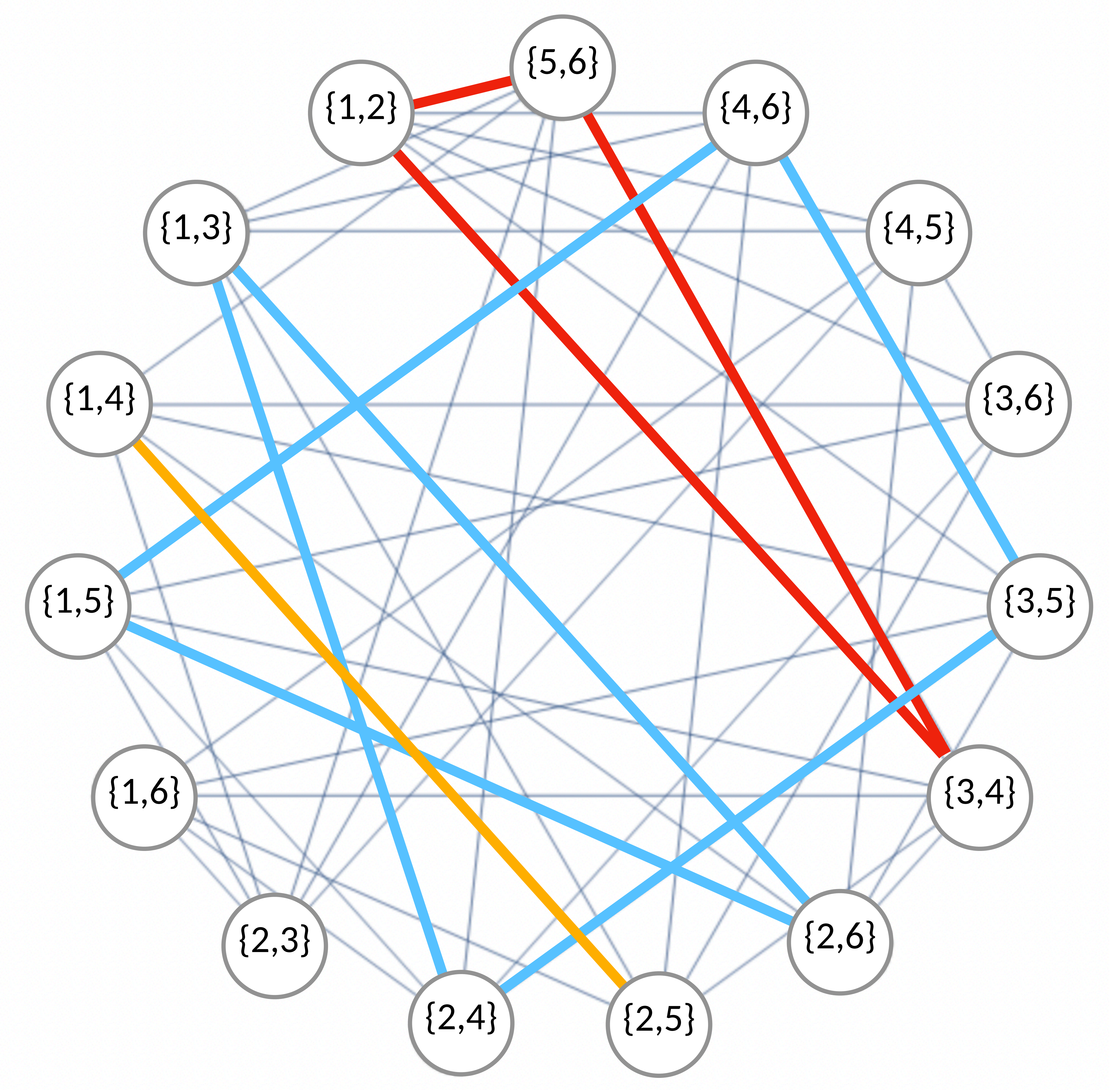
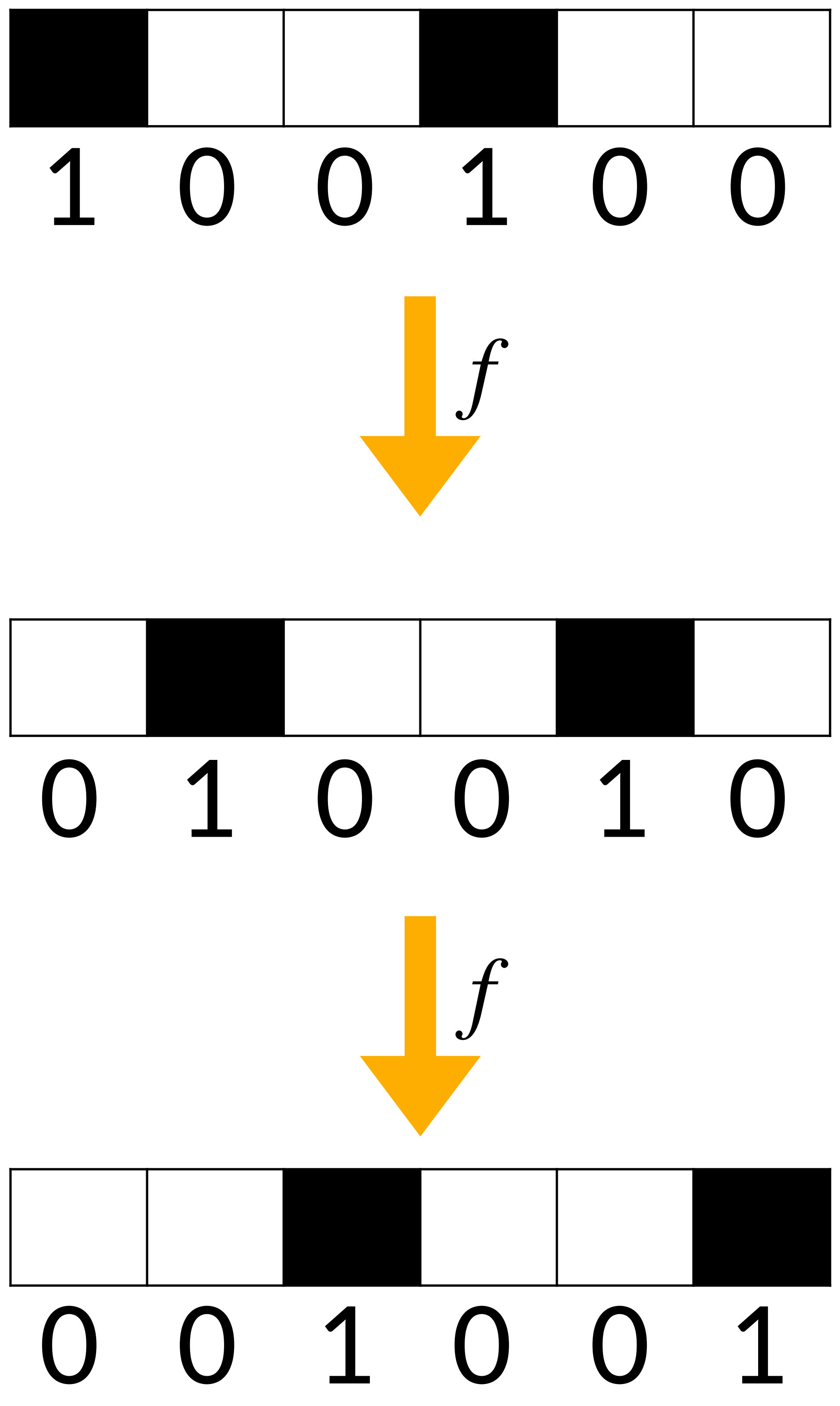




$K(6, 2)$

Example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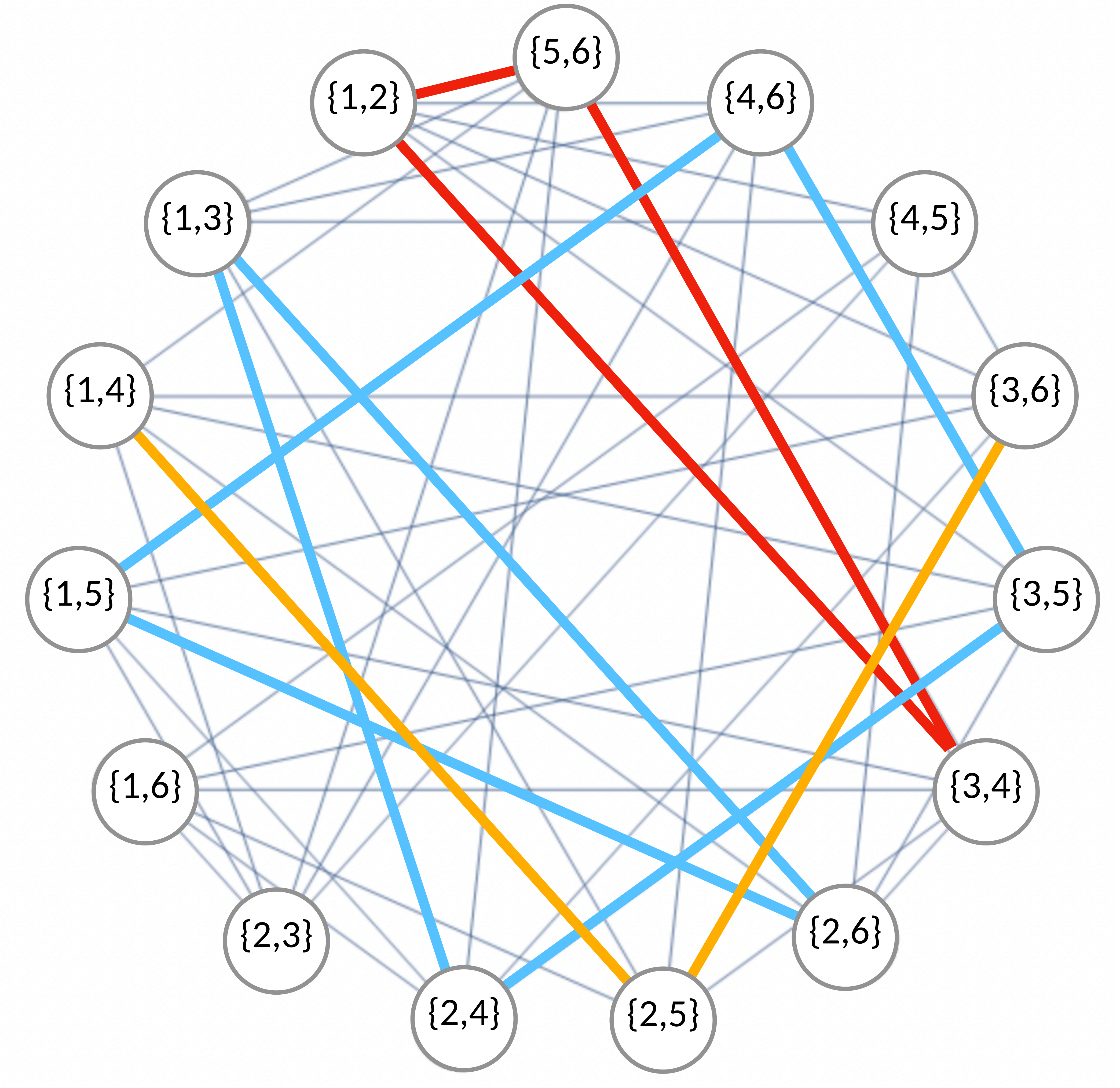
0 1 0 0 1 0
f
0 0 1 0 0 1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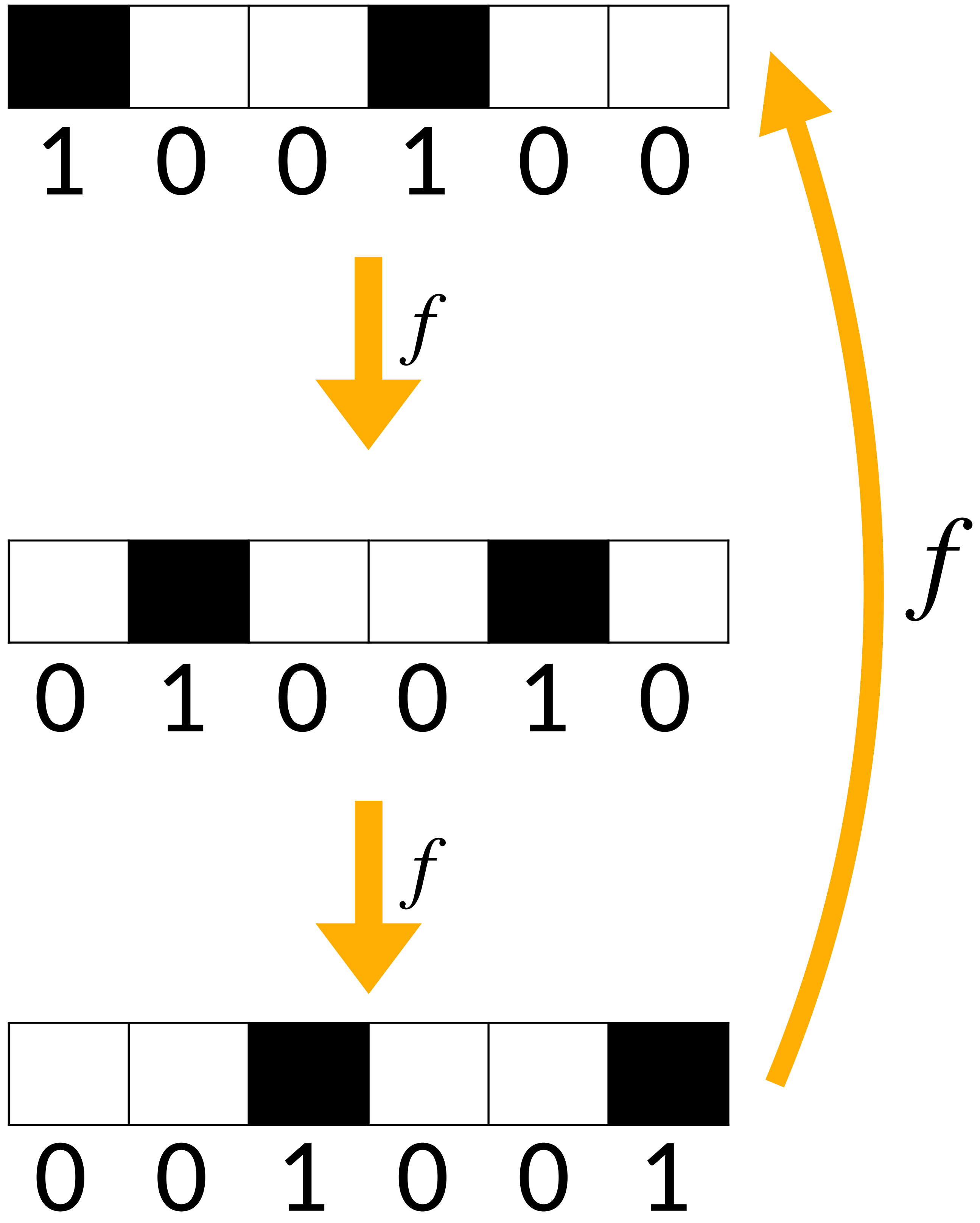
$K(6, 2)$

Example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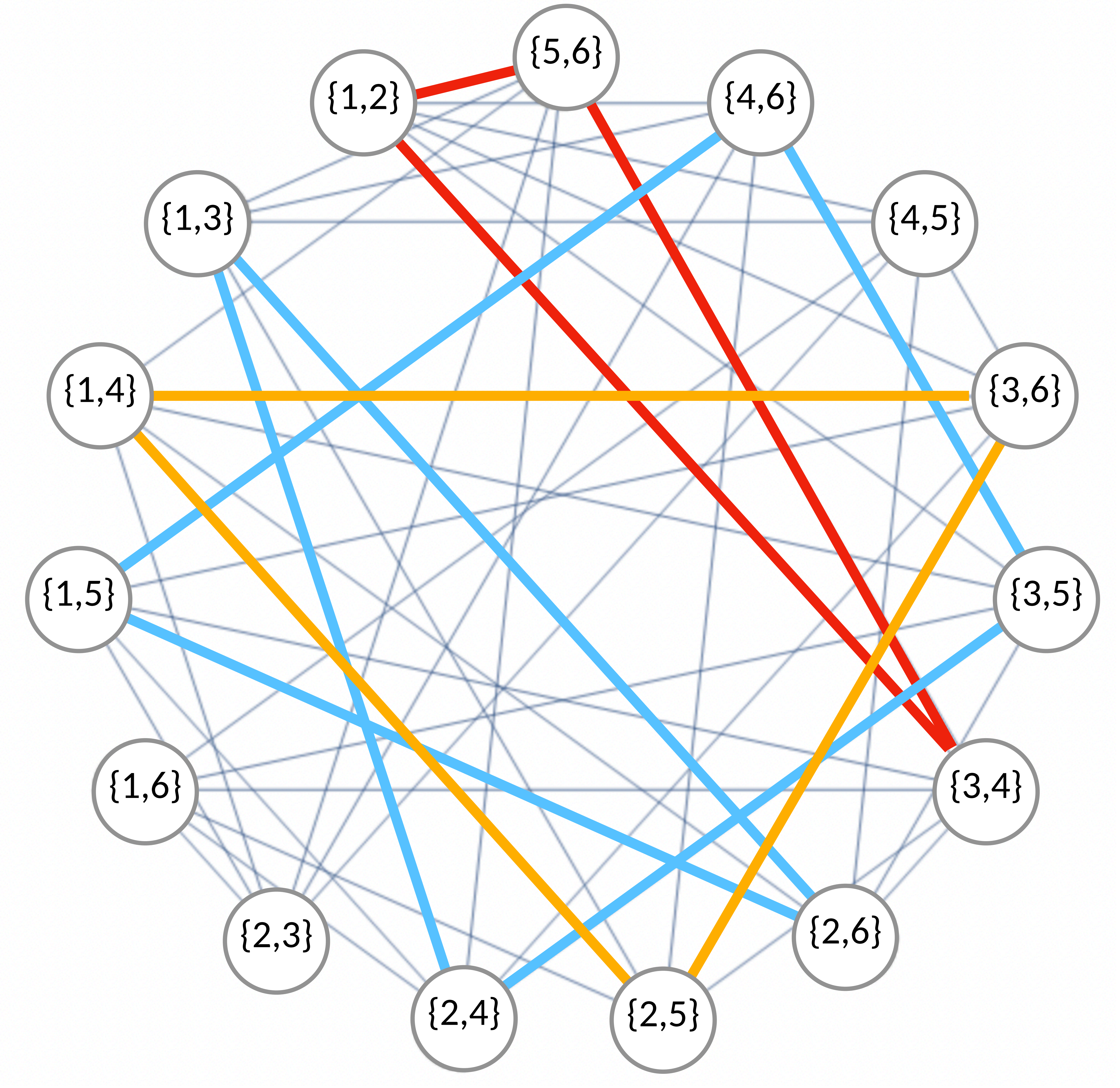
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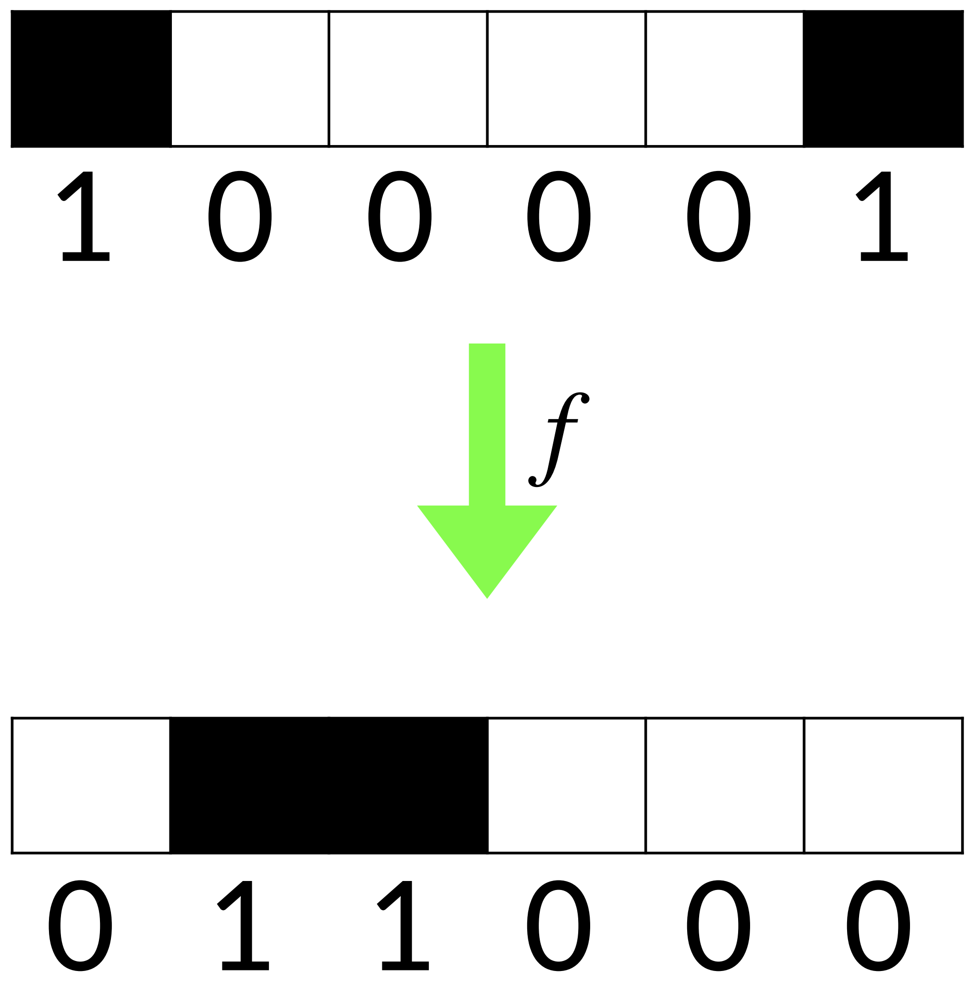
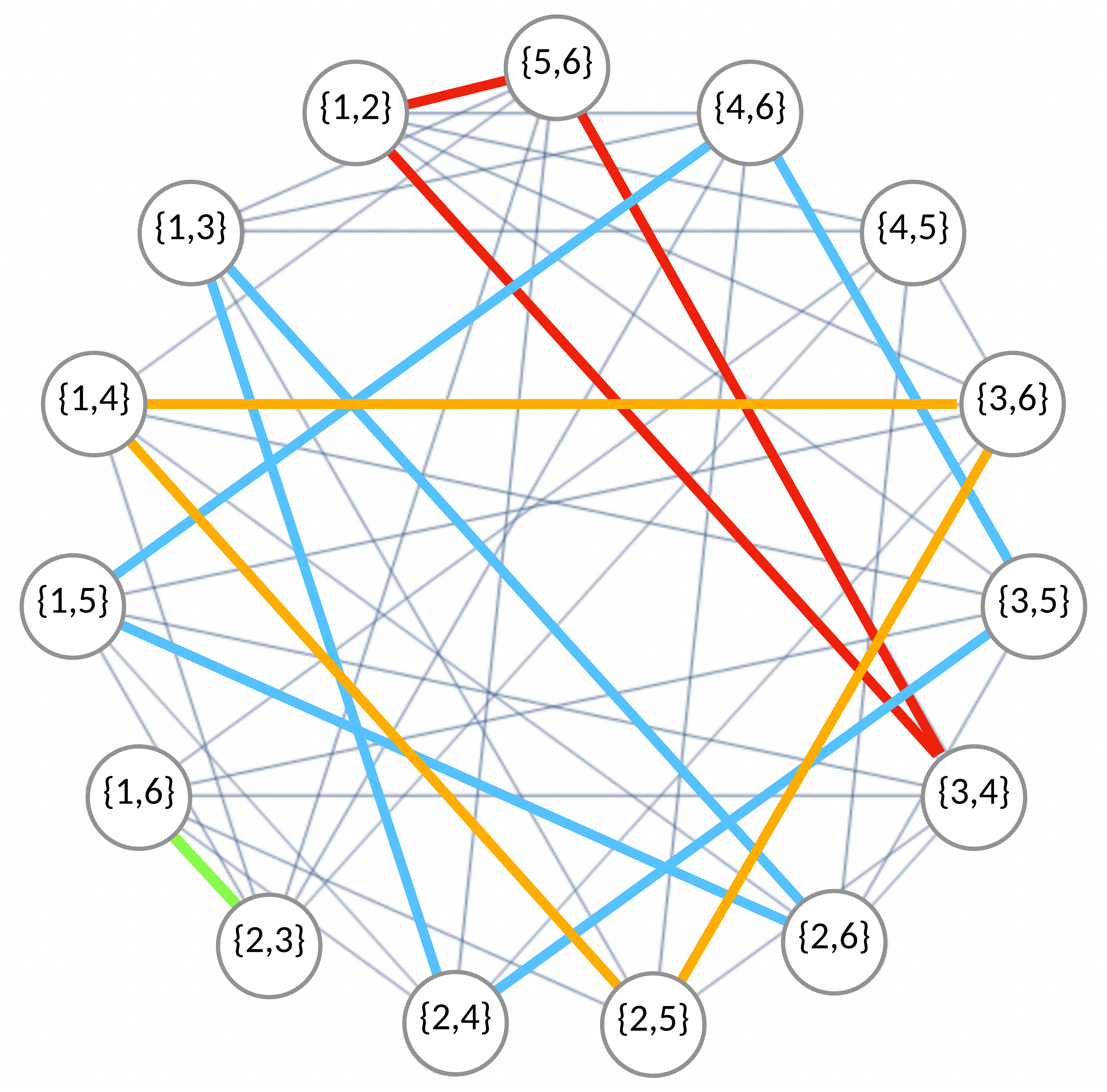
$K(6, 2)$

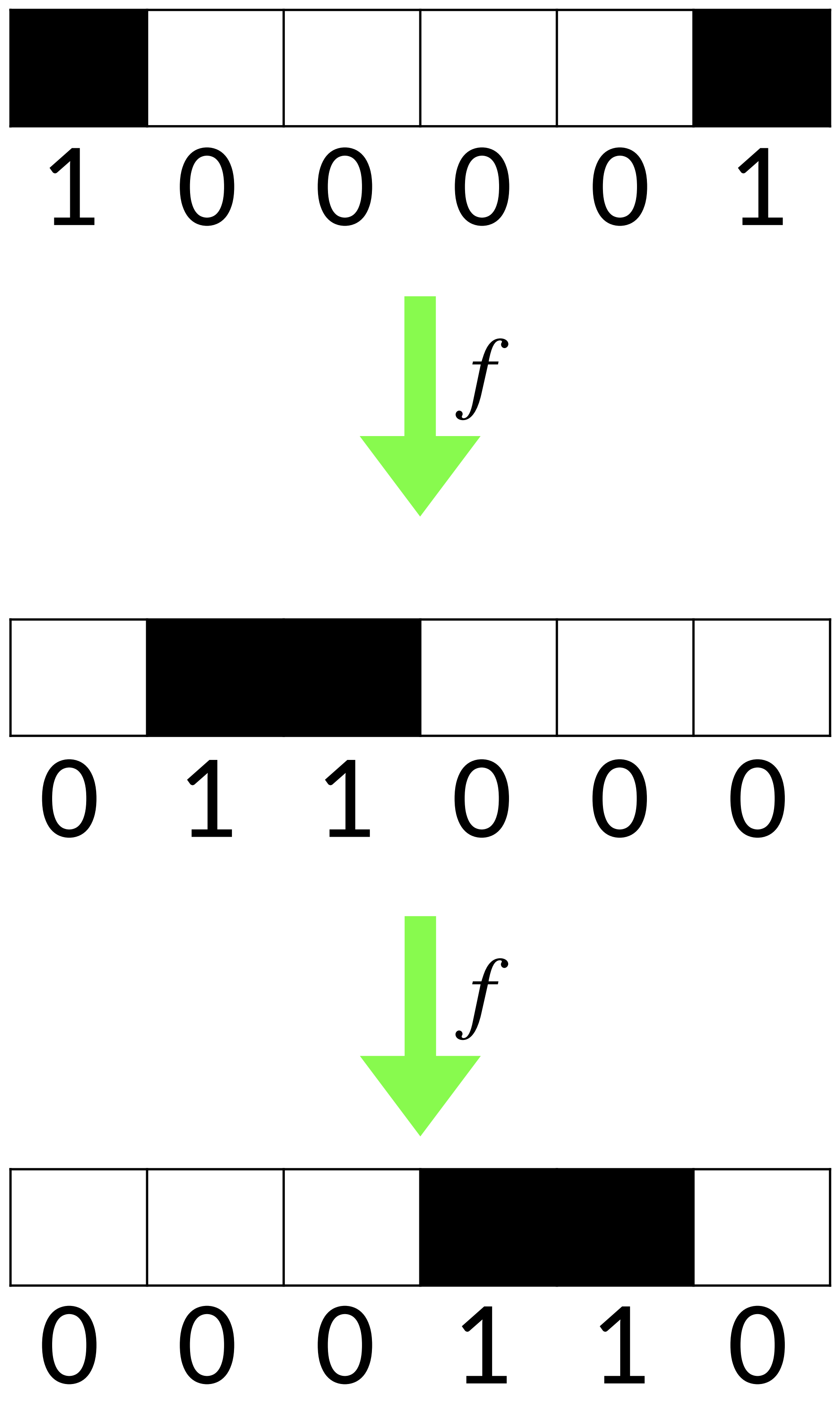
Example





$K(6, 2)$

Example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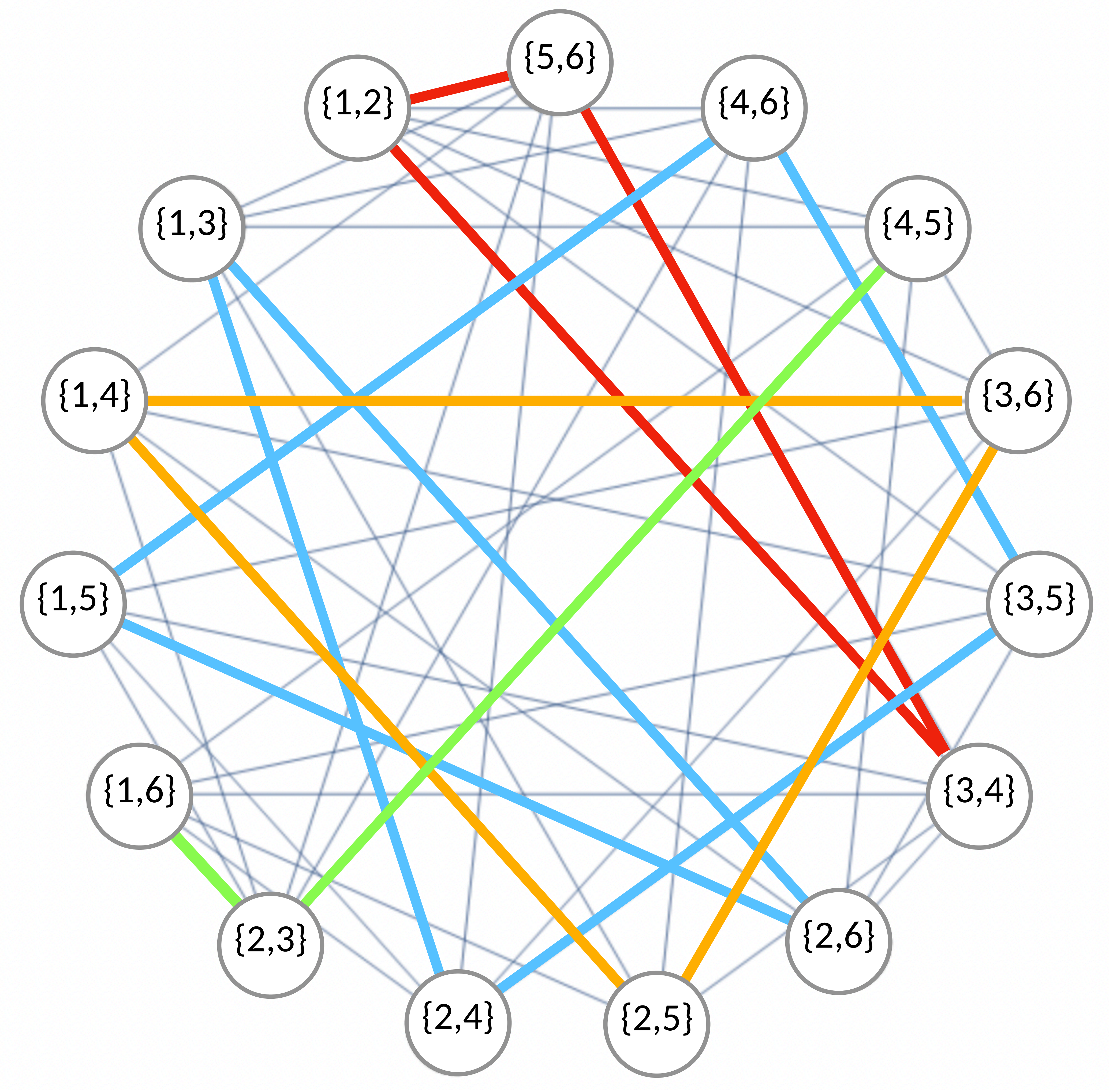
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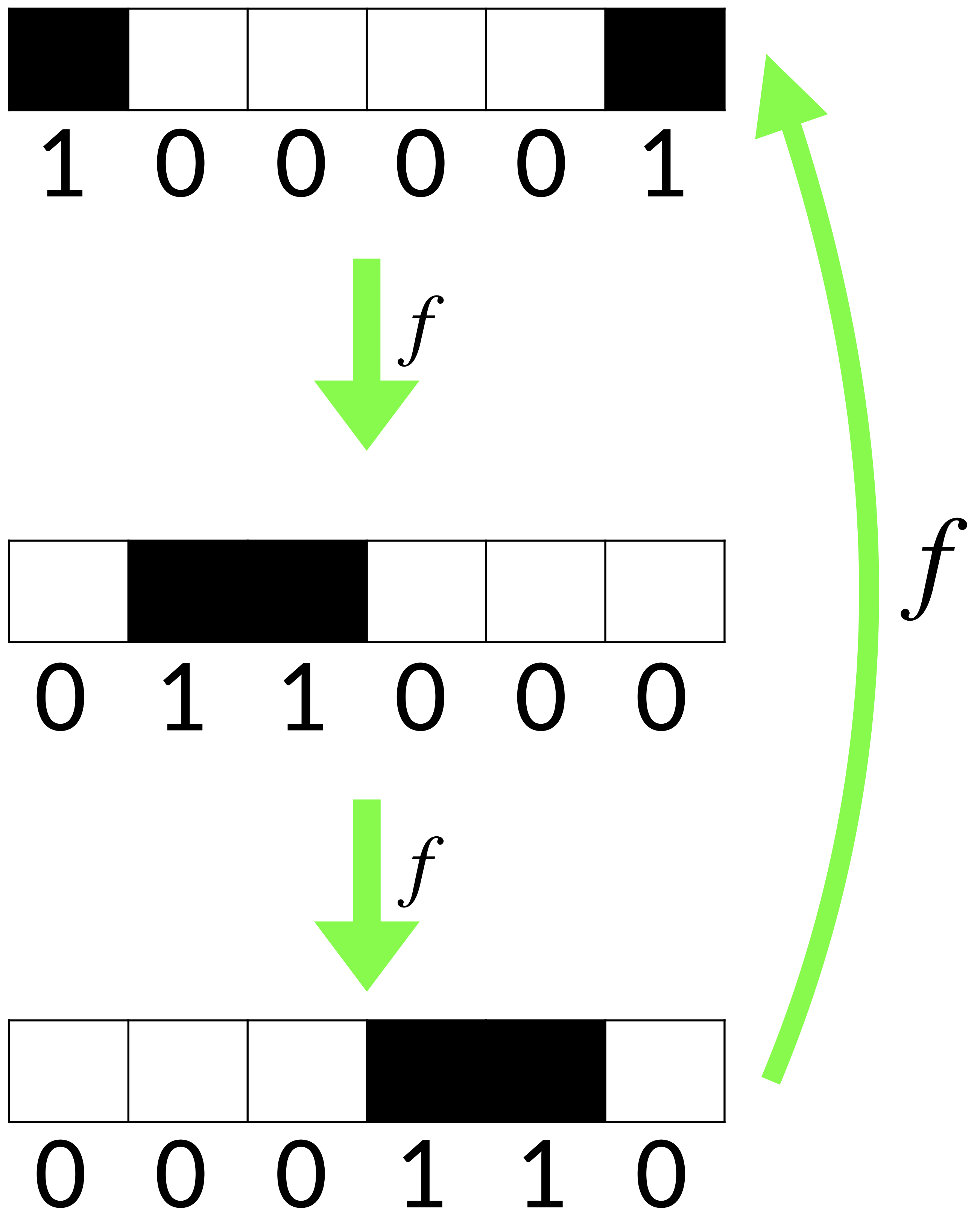
$K(6, 2)$

Example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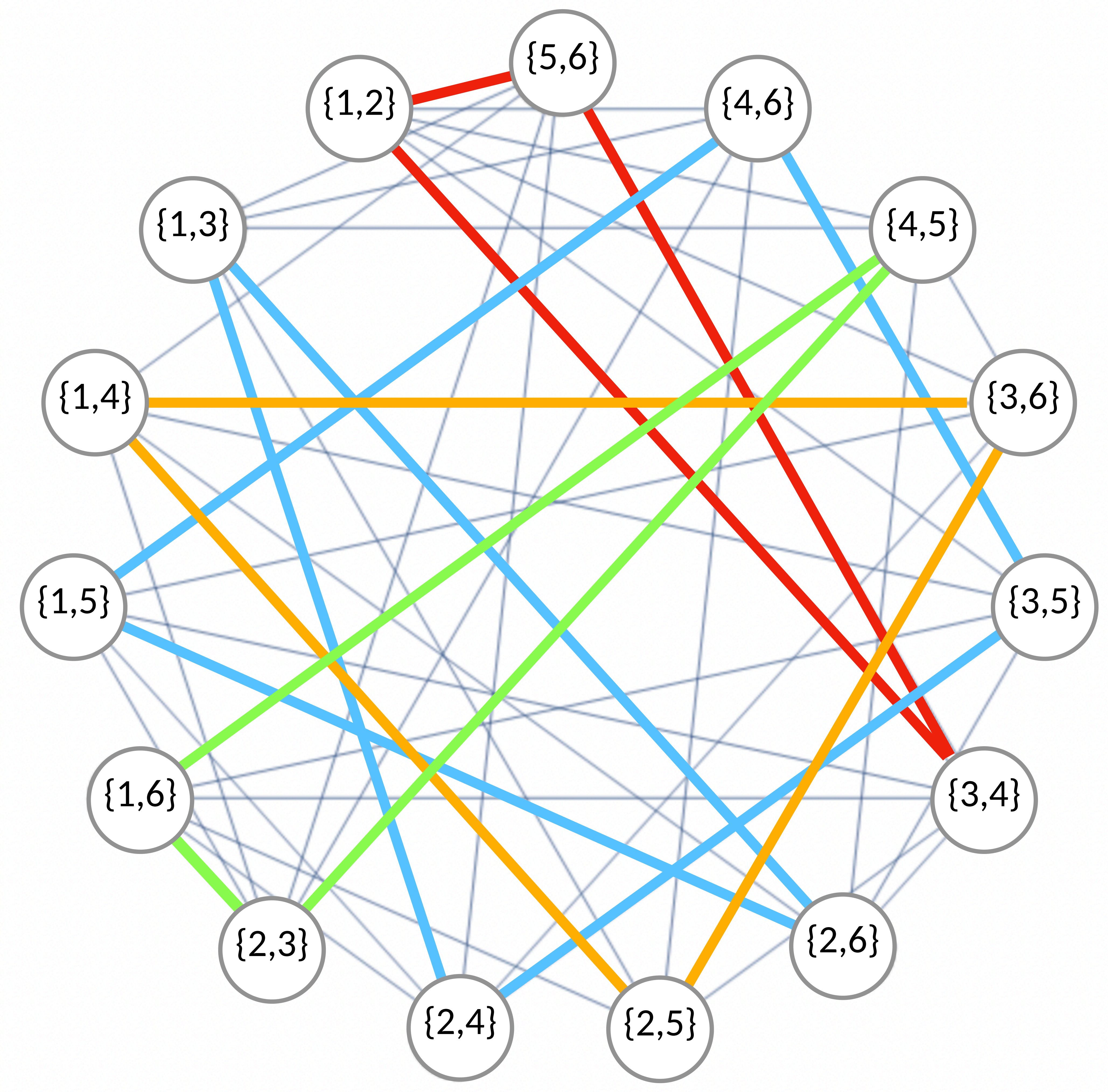
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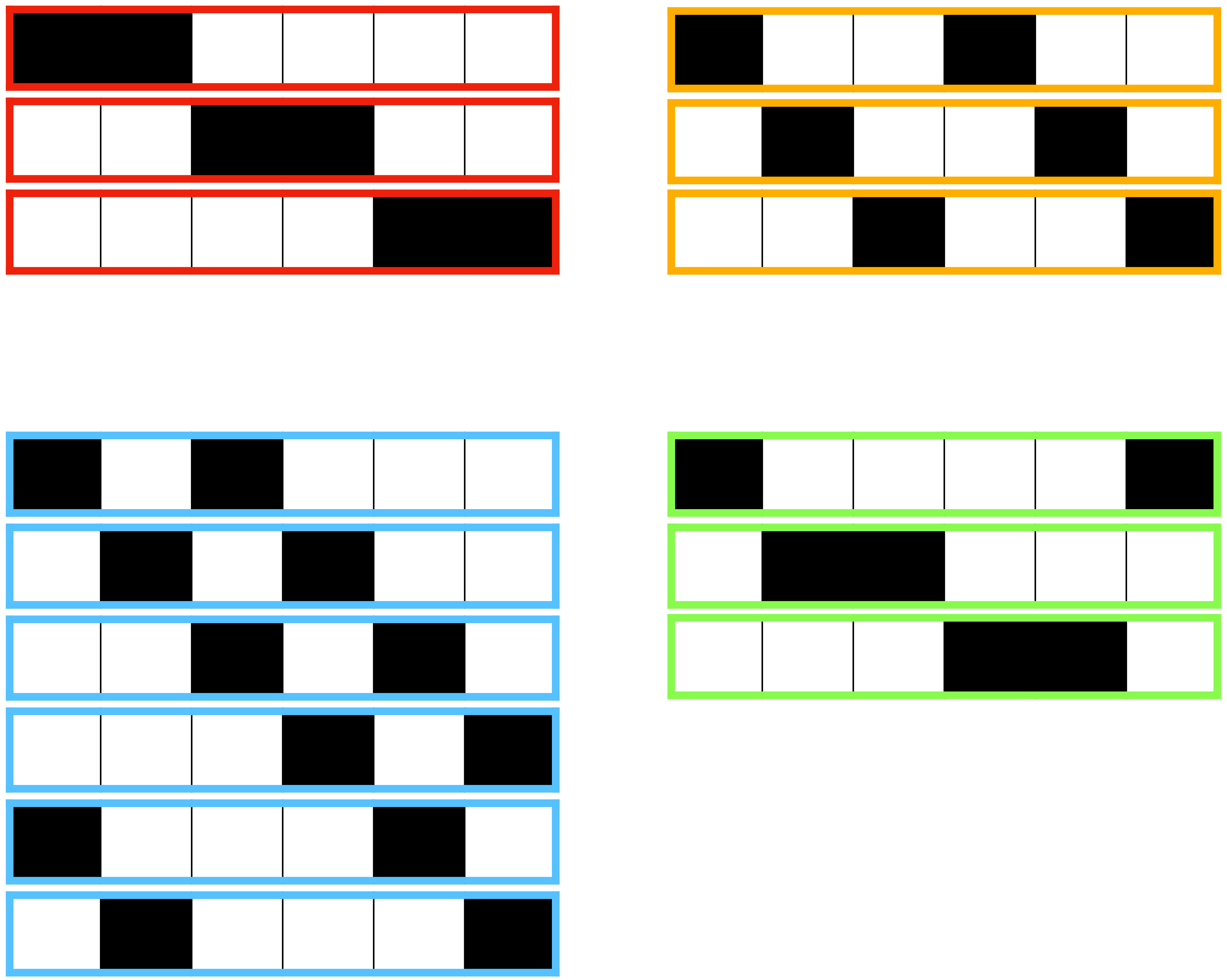
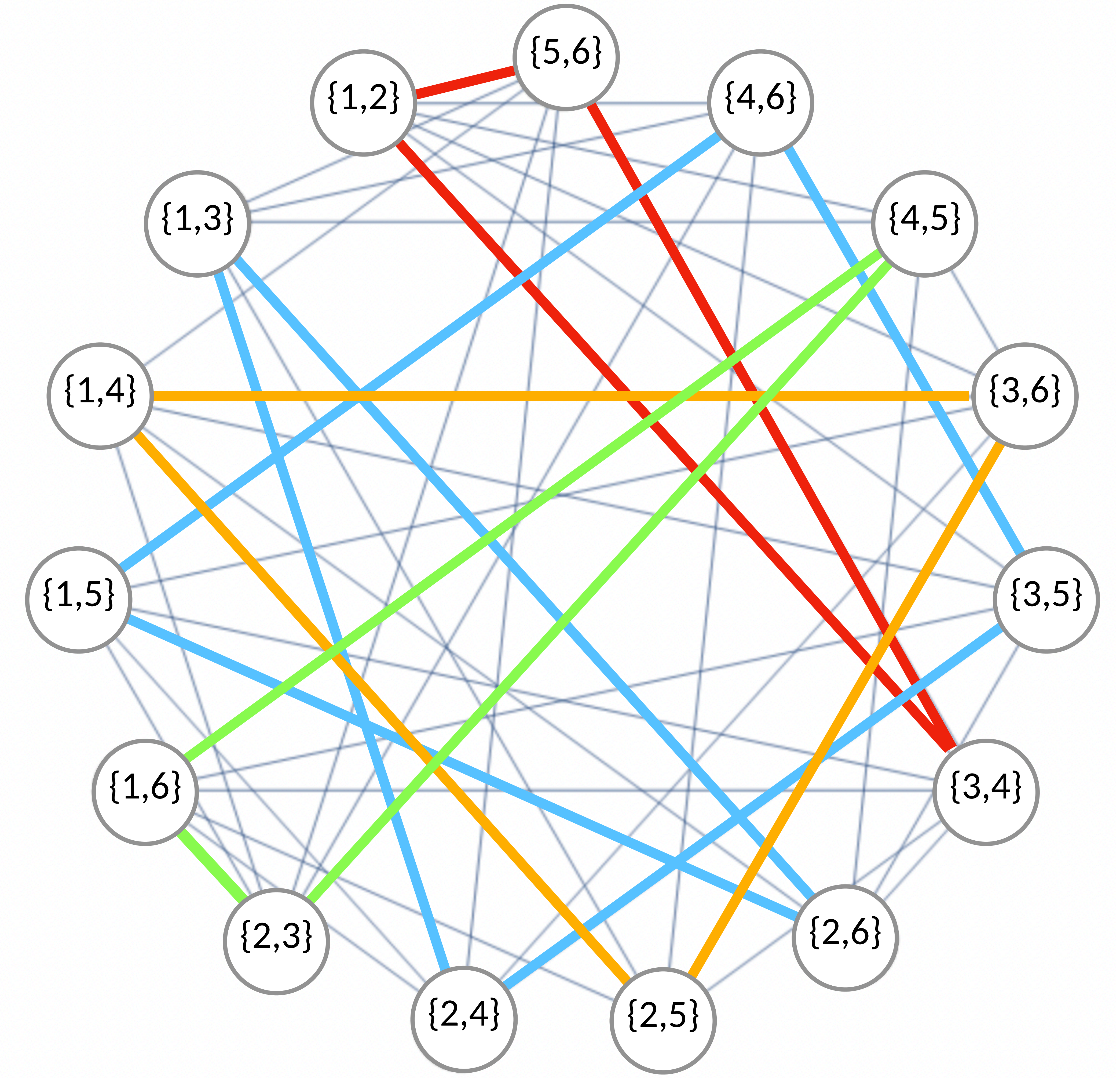
$K(6, 2)$

Cycle factors

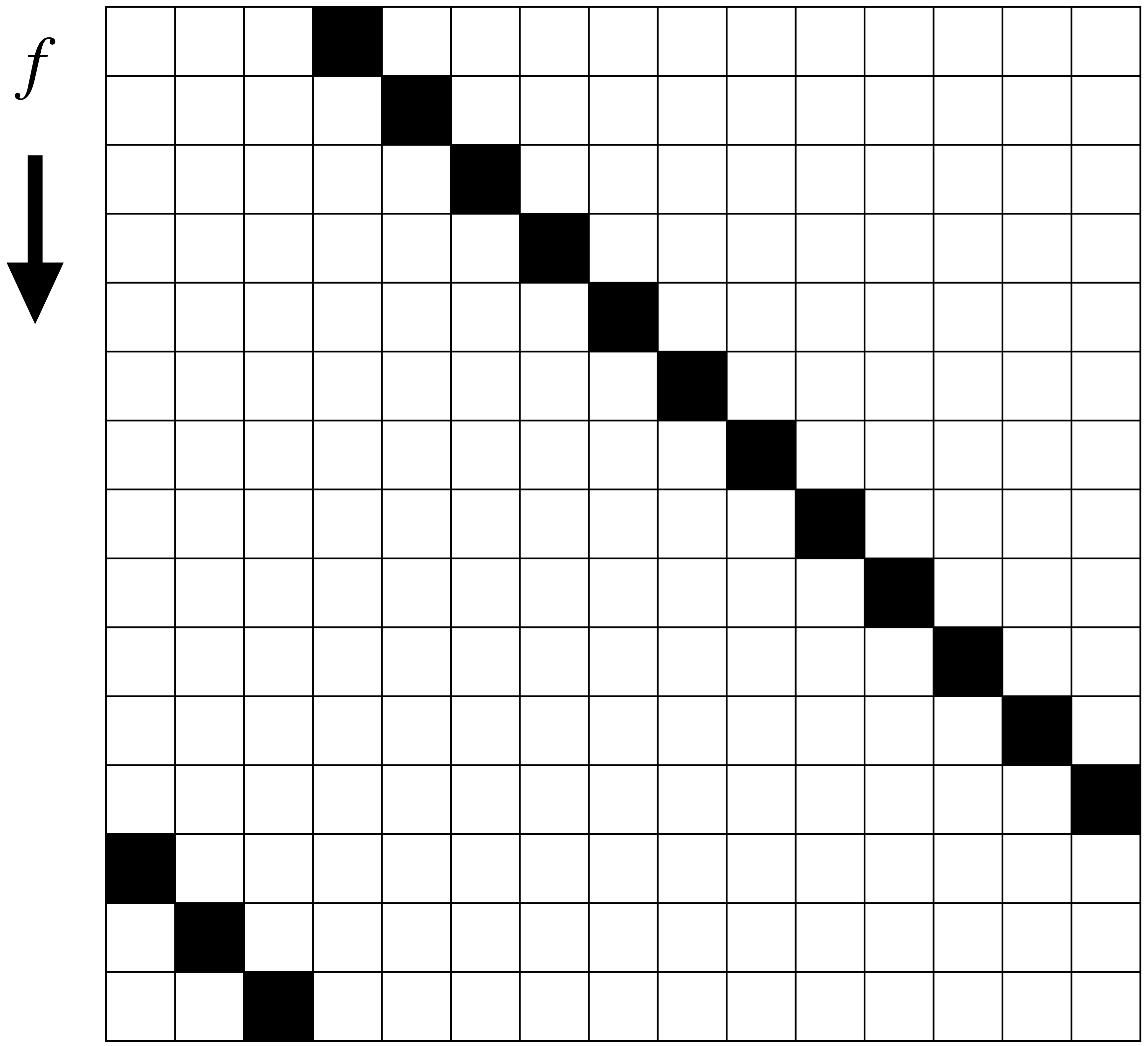




$K(6, 2)$

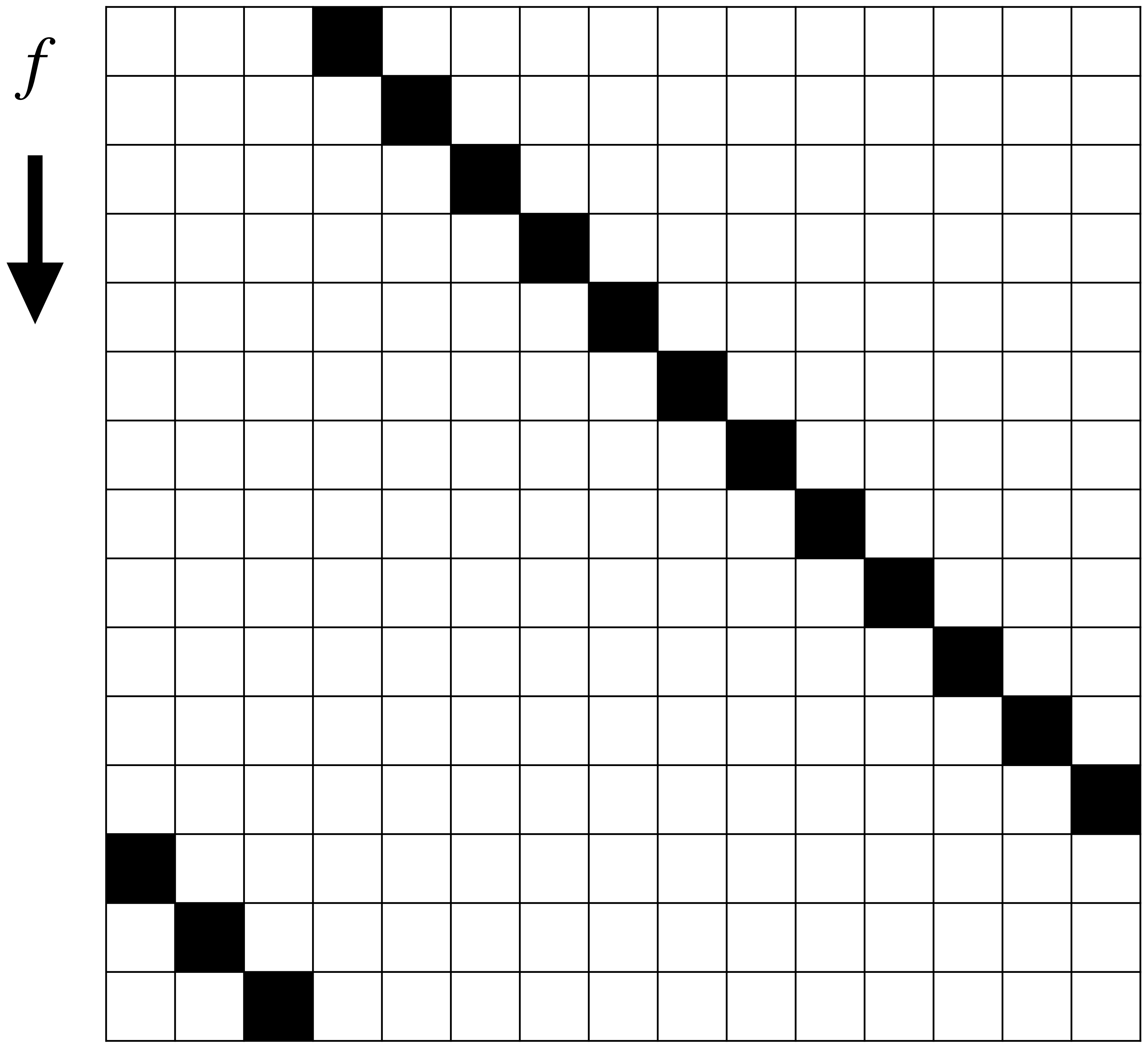
Analysis of cycles

$K(15, 1)$

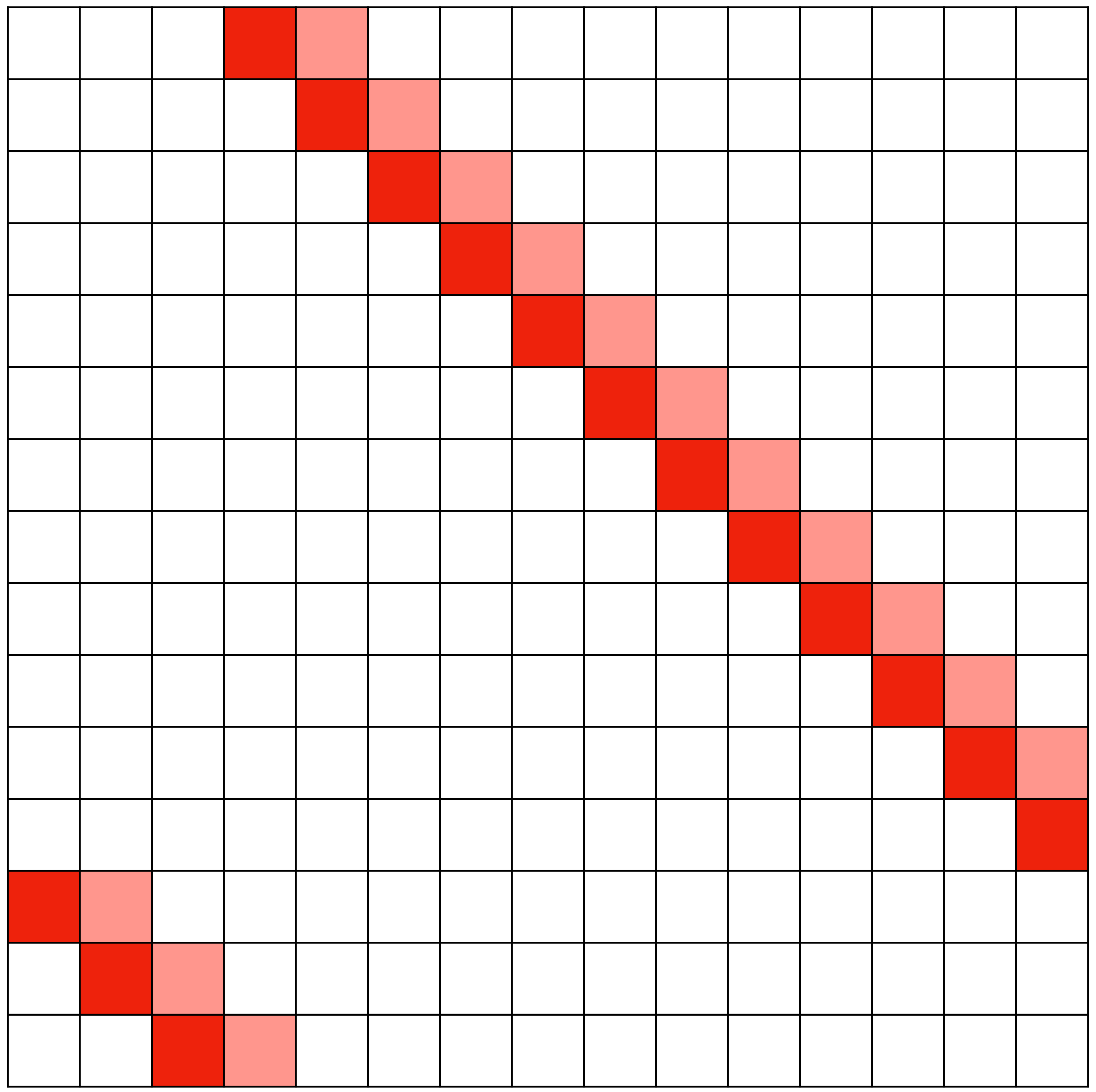


Analysis of cycles

$K(15, 1)$

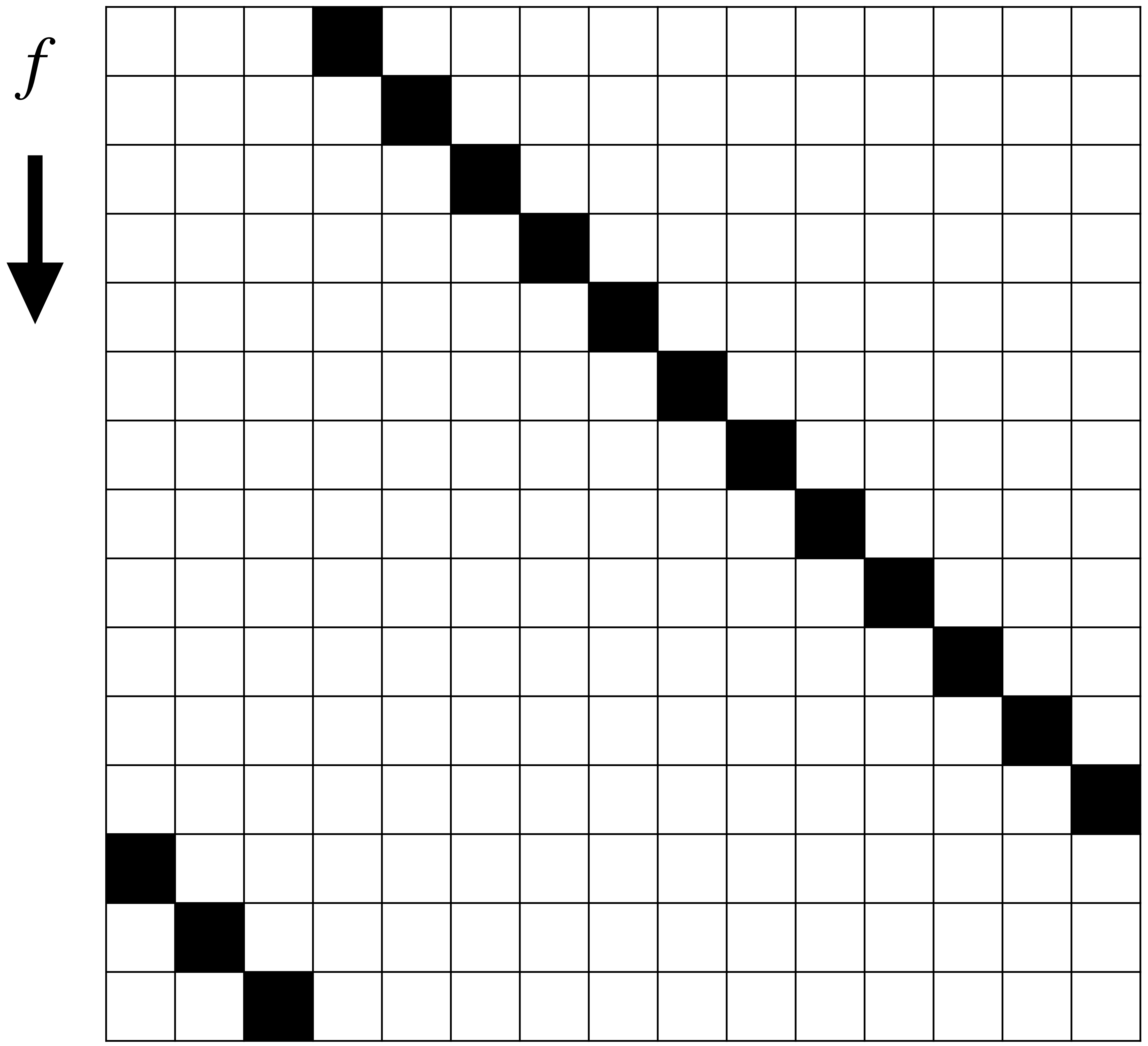


$K(15, 1)$

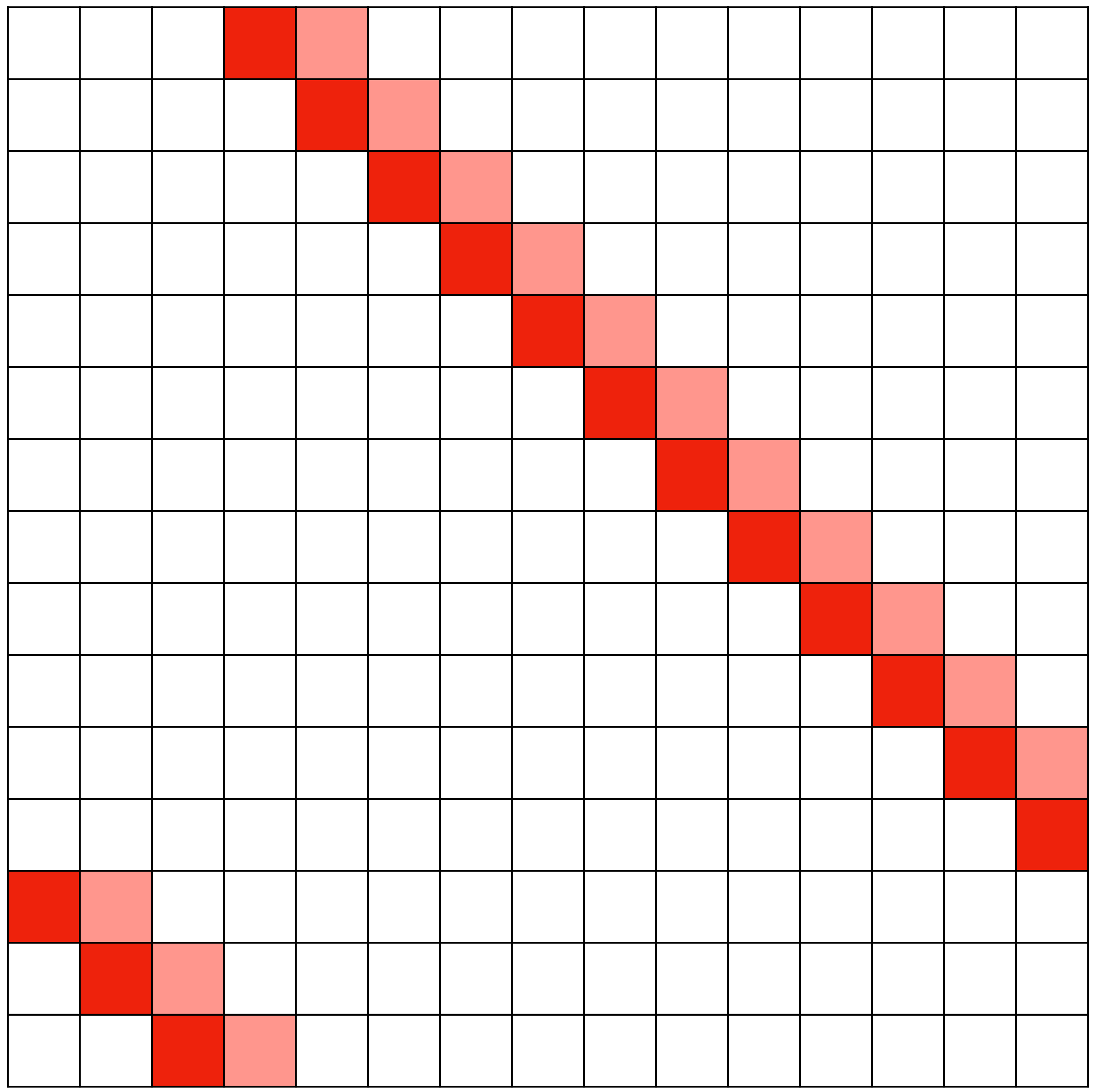


Analysis of cycles

$K(15, 1)$

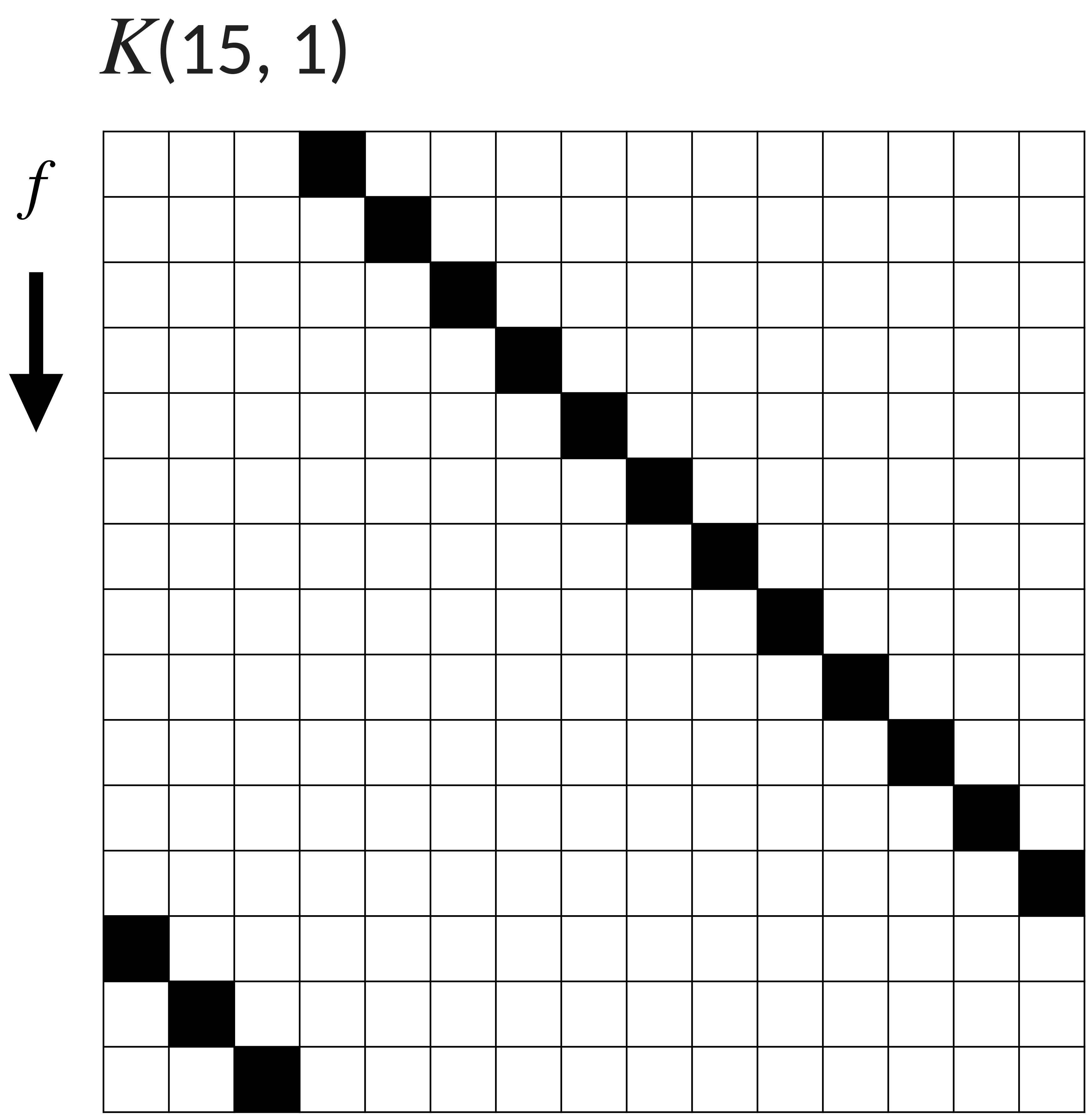


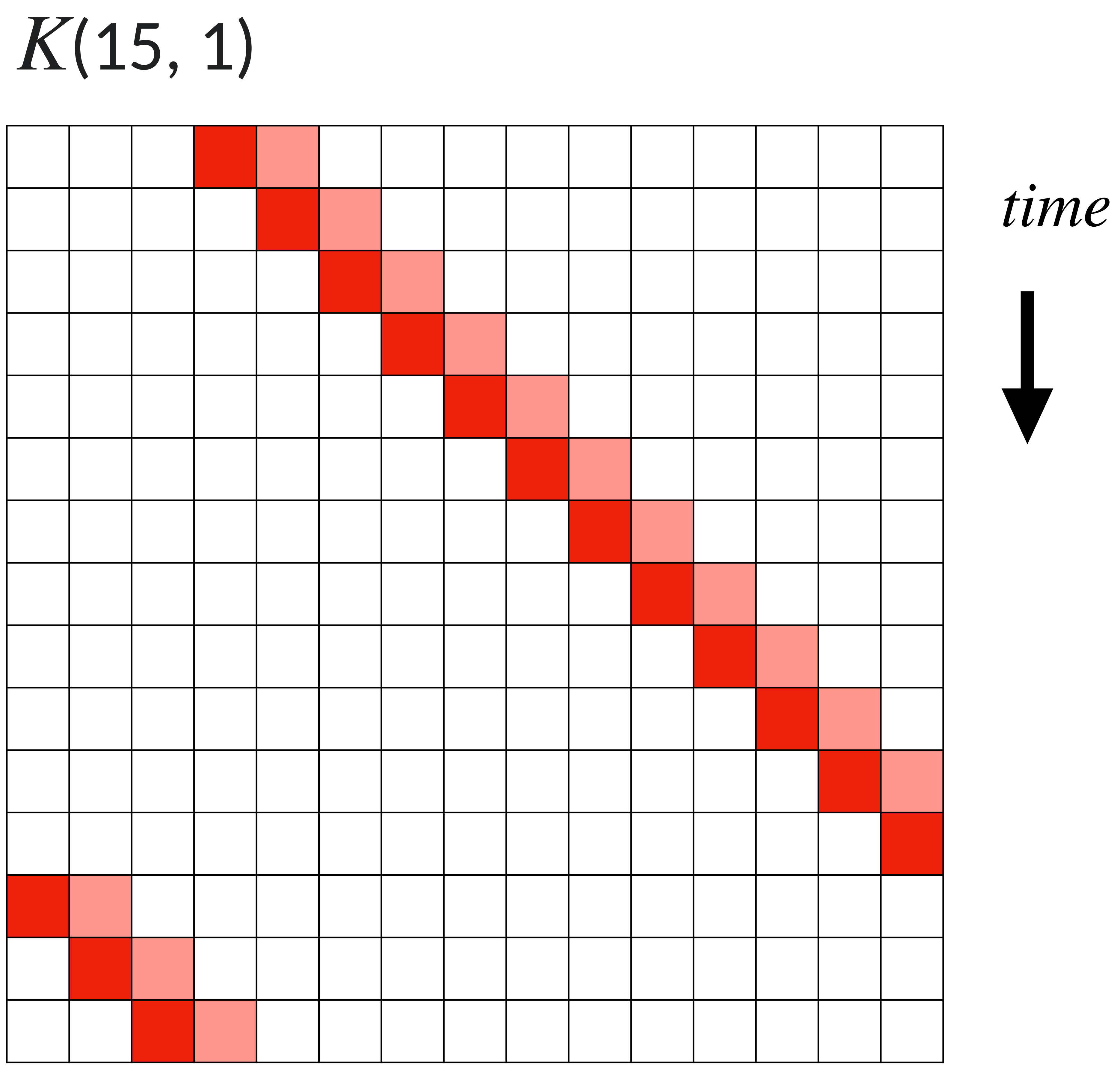
$K(15, 1)$



- two matched bits form a glider

Analysis of cycles

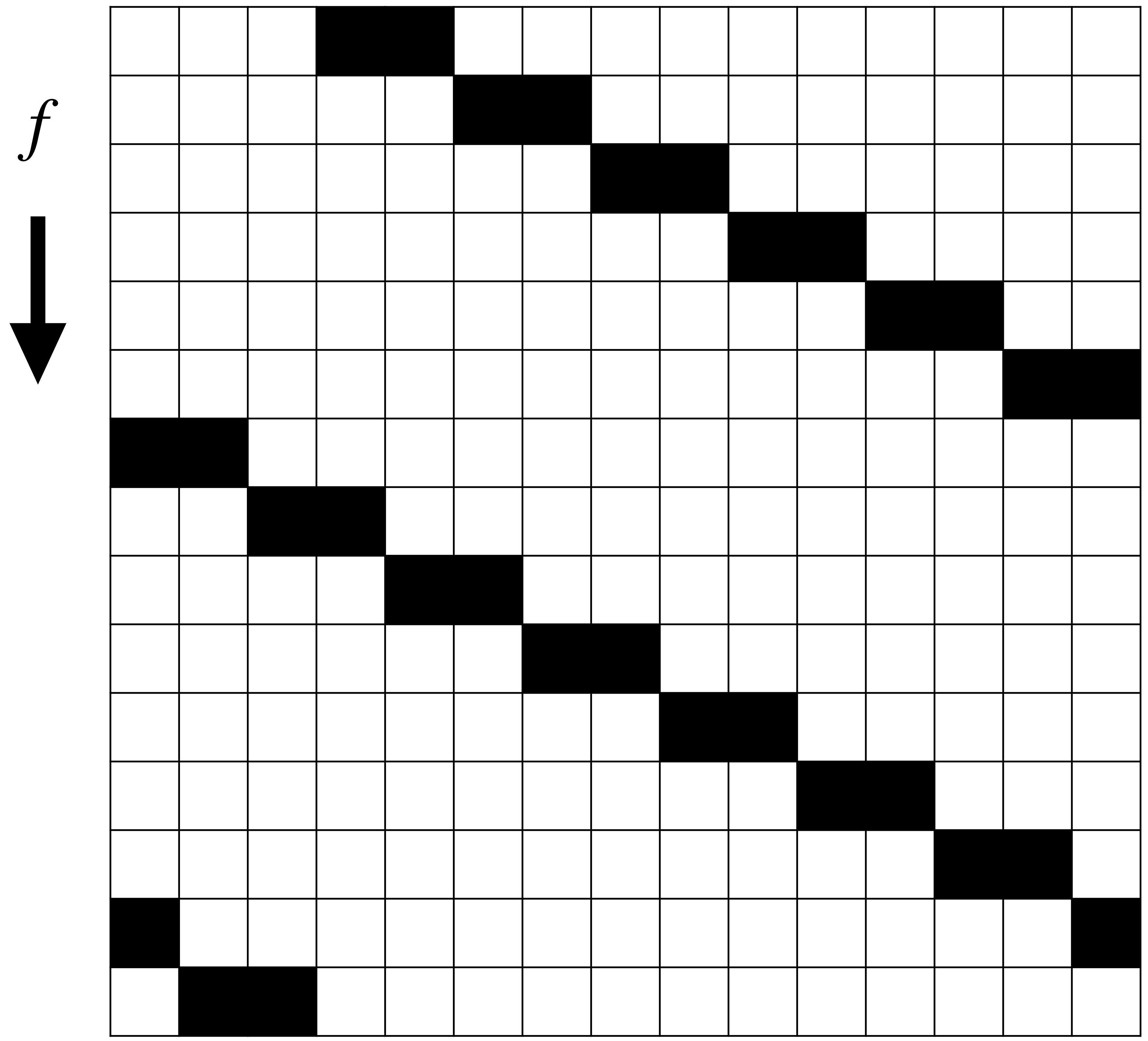




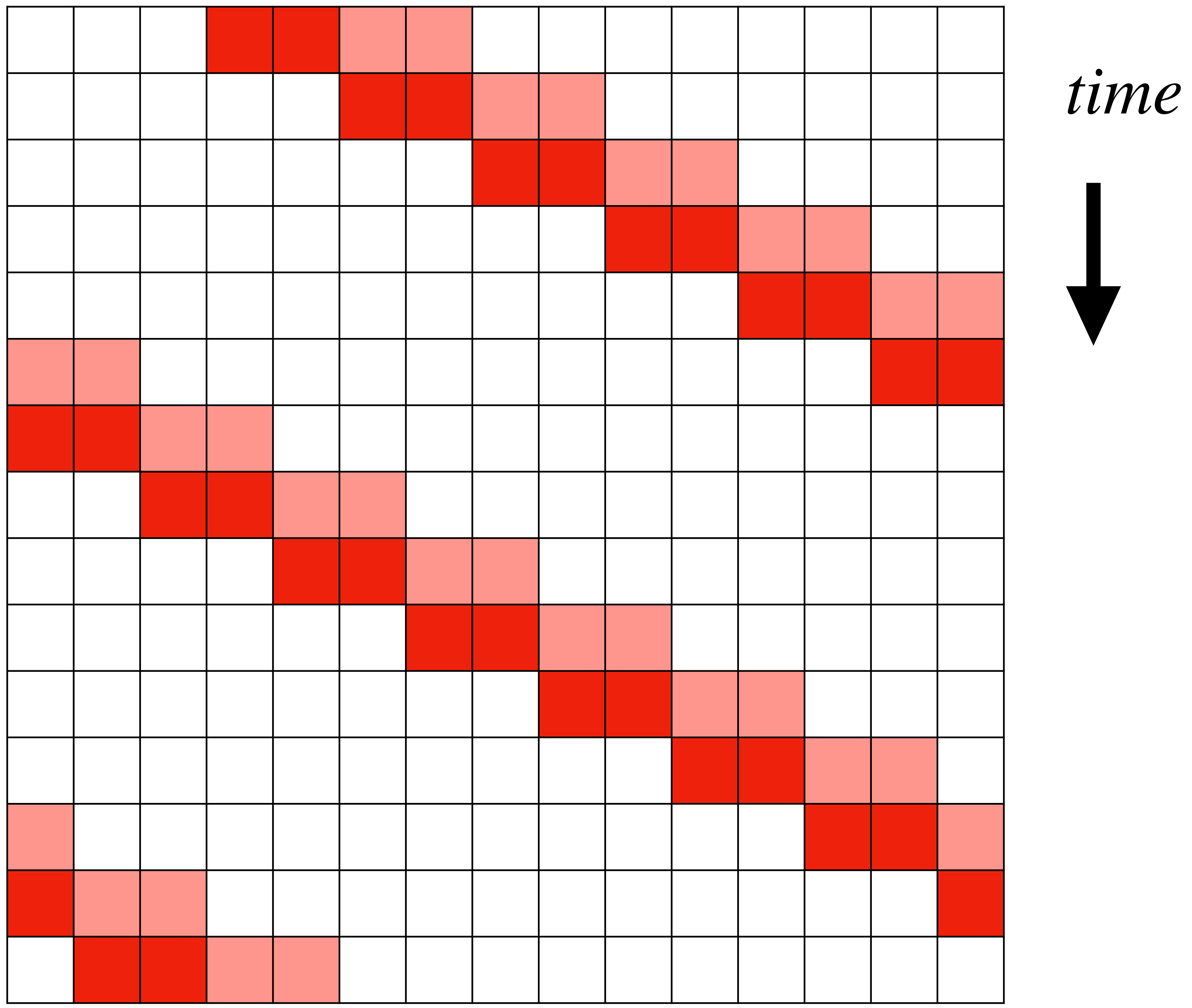
- two matched bits form a glider
- glider **moves** by one unit per step

Analysis of cycles

$K(15, 2)$



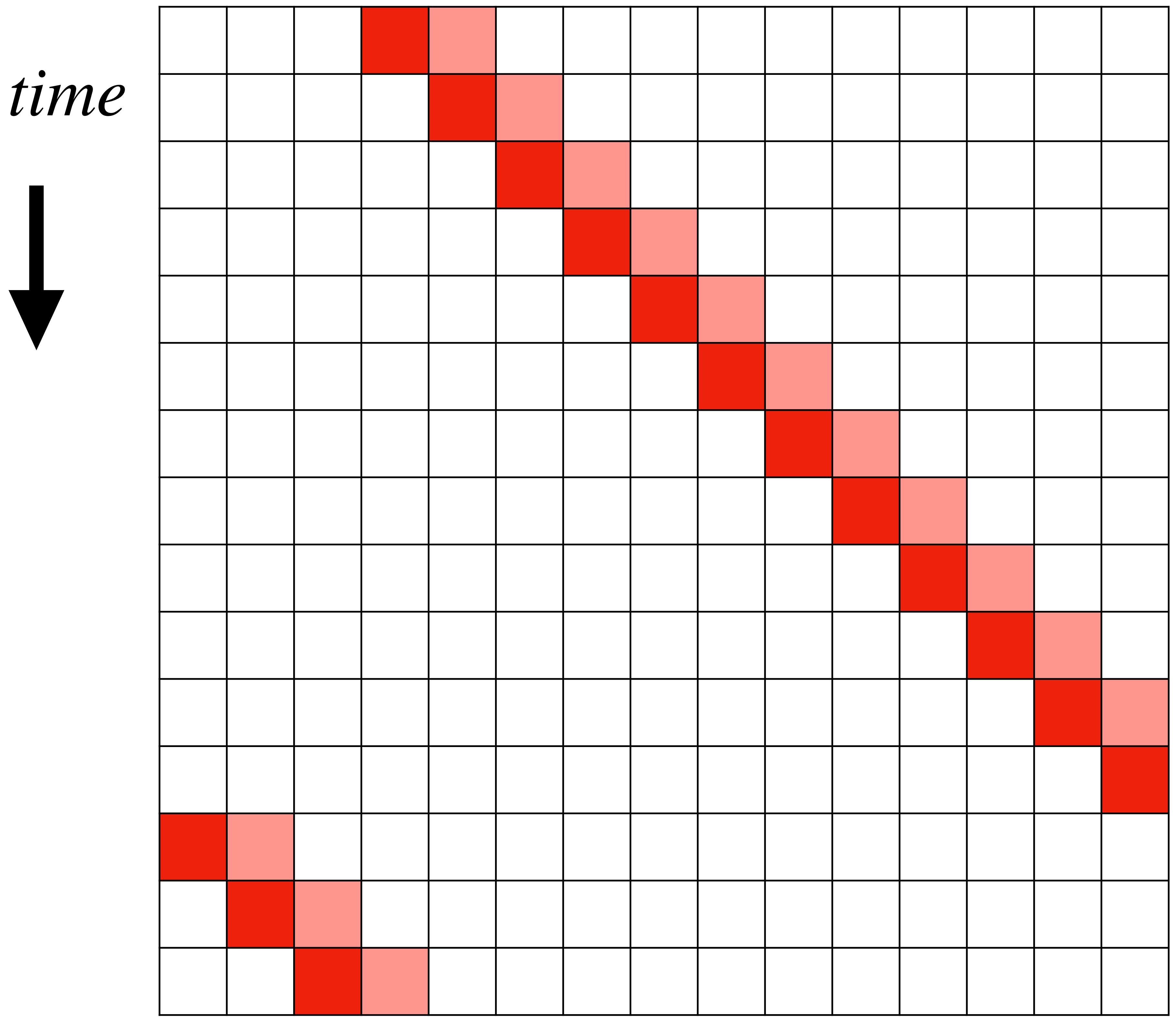
$K(15, 2)$



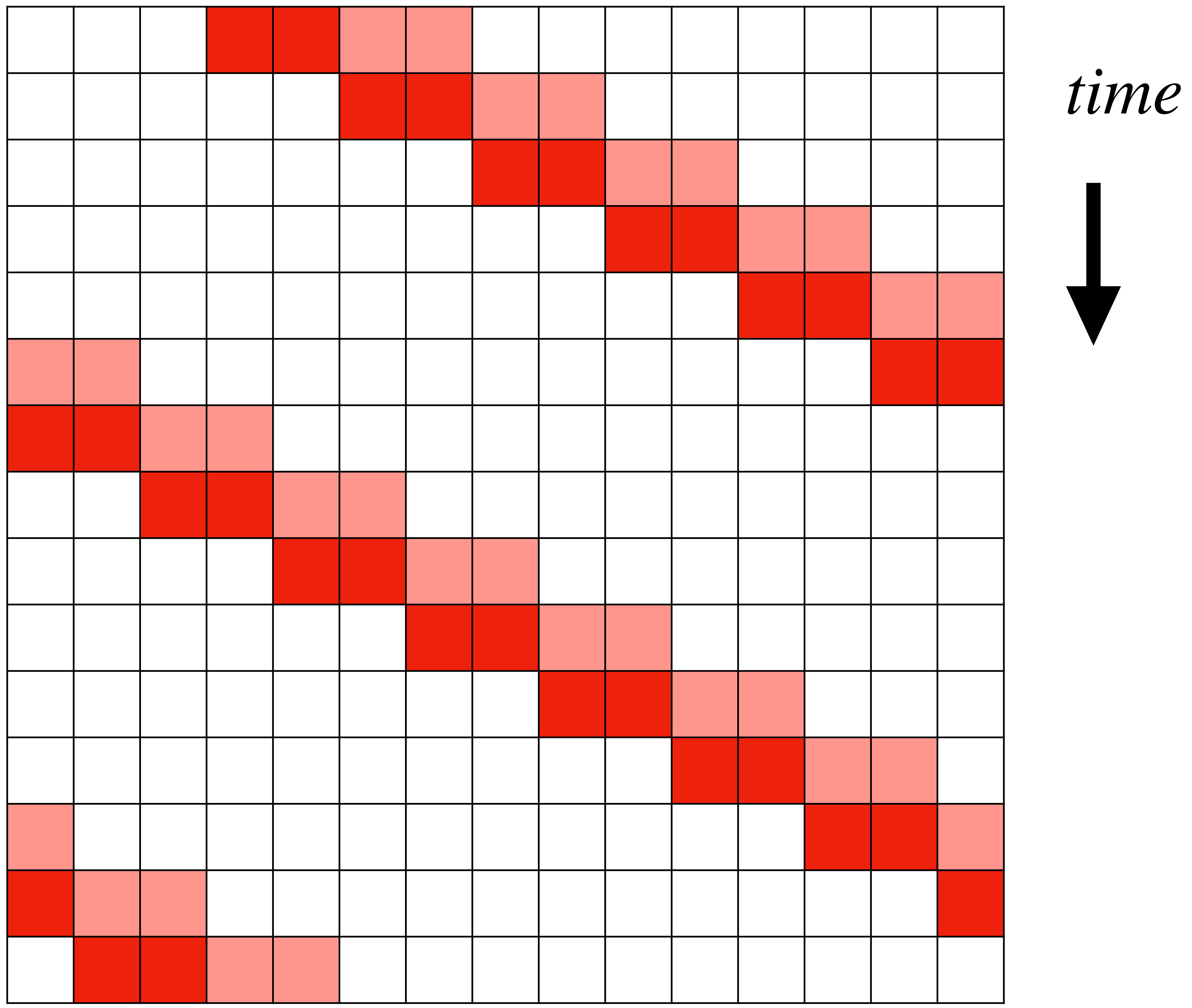
- four matched bits form a glider
- glider **moves** by two units per step

Gliders

$K(15, 1)$

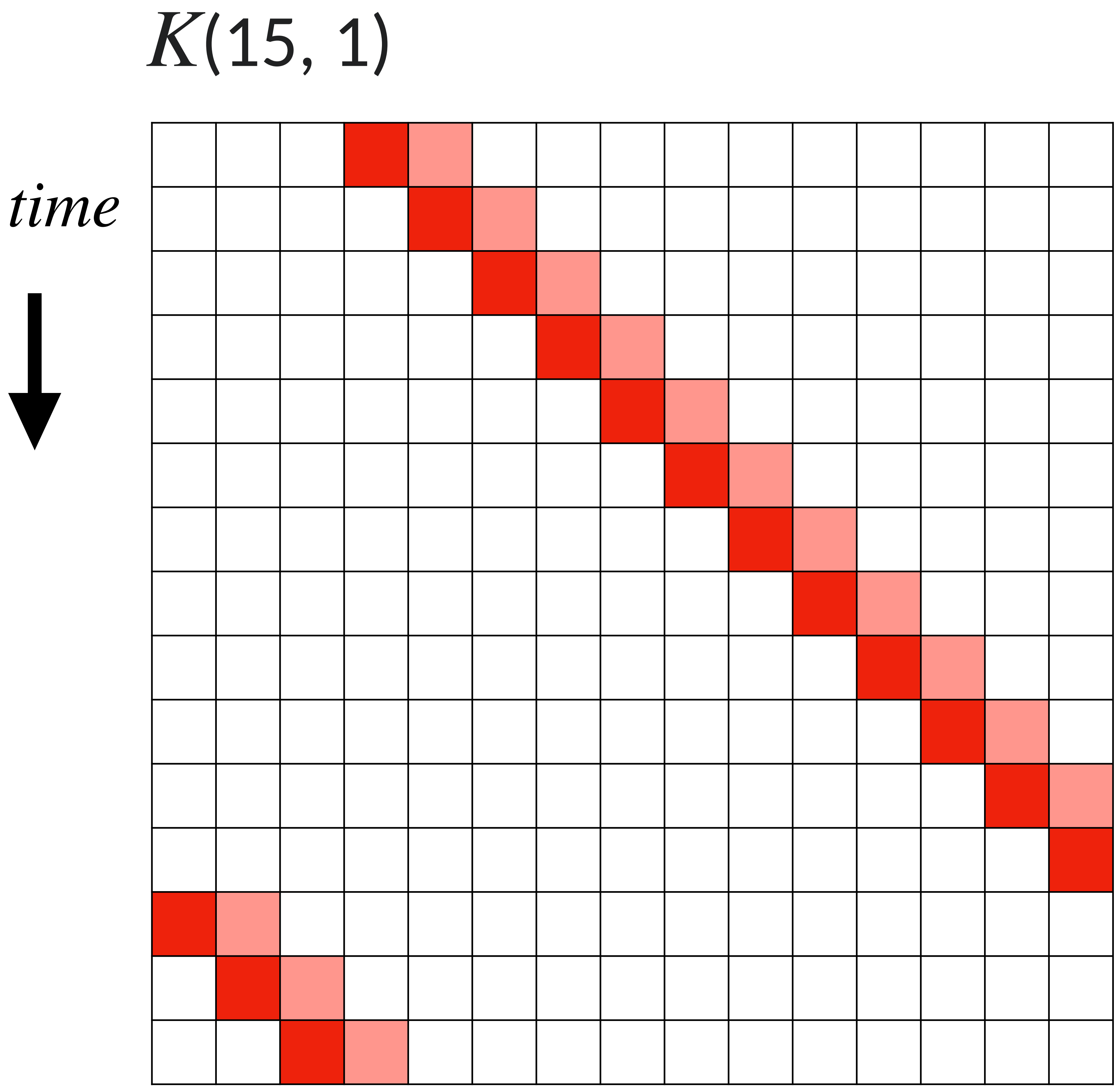


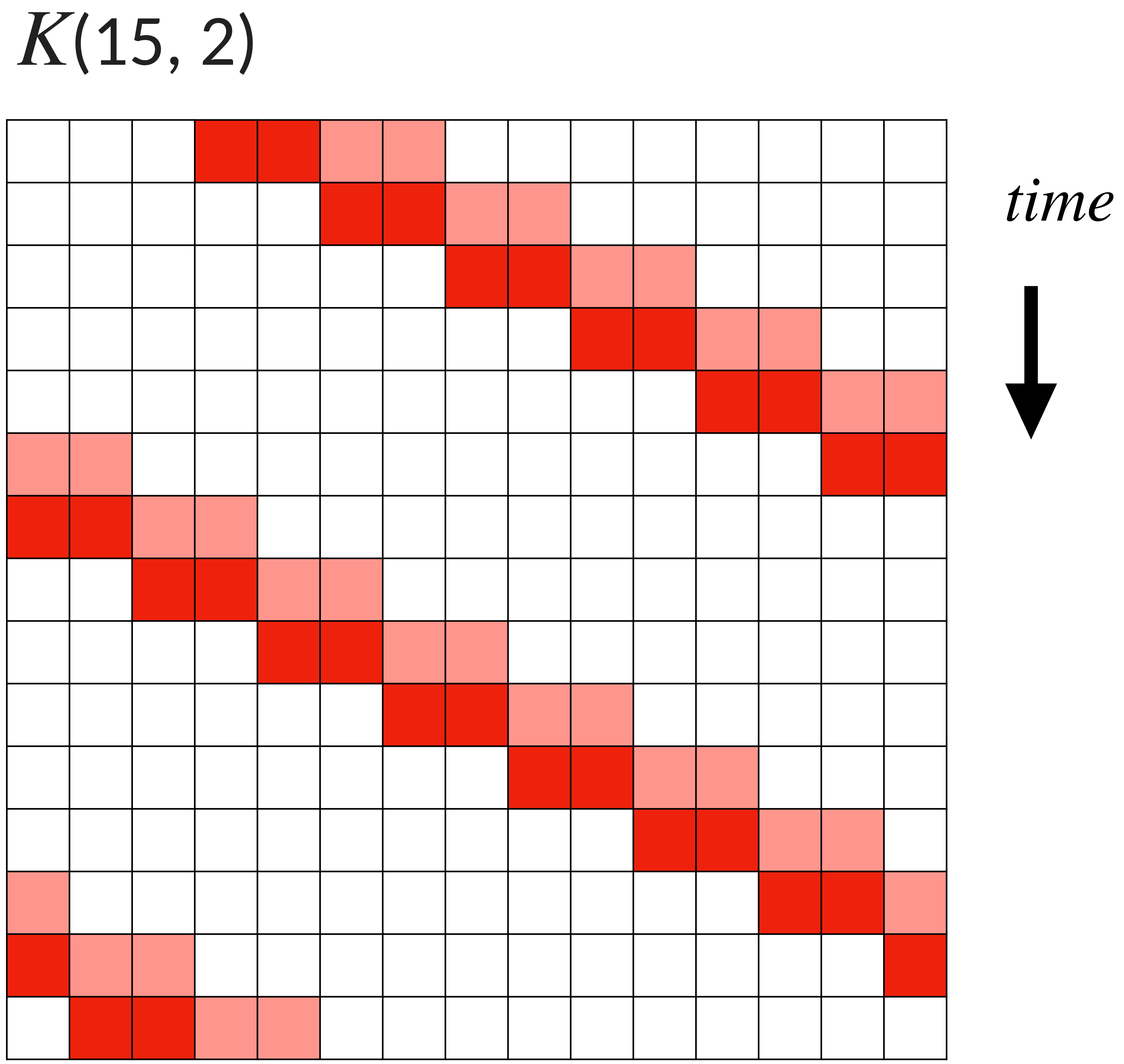
$K(15, 2)$



- glider = set of matched 1s and 0s

Gliders

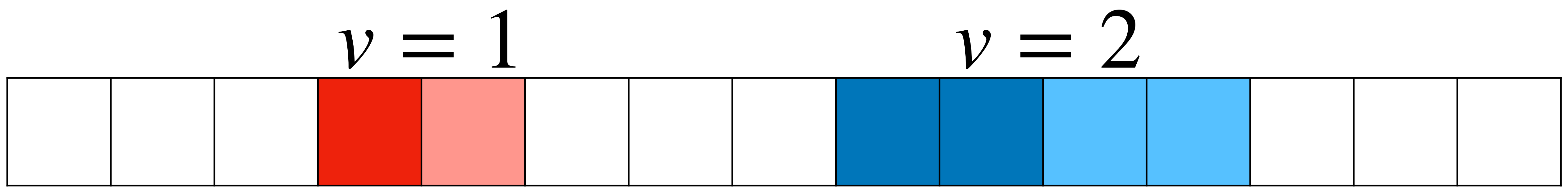




- glider = set of matched 1s and 0s
- speed (v) = numbers of 1s = number of 0s

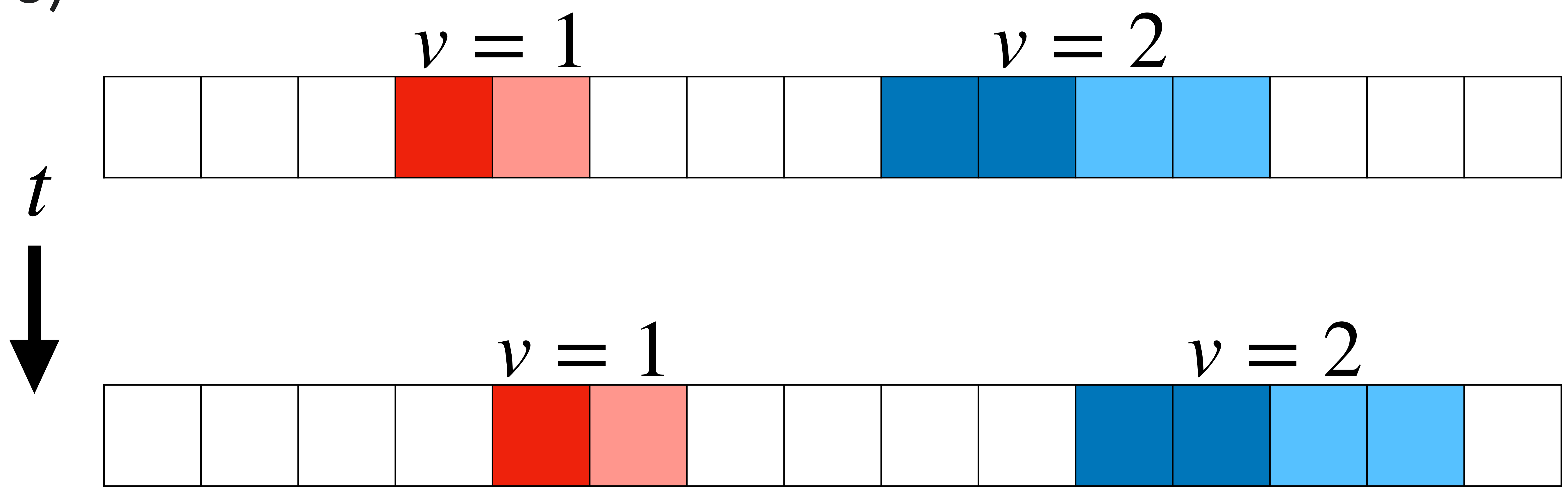
Motion of gliders

$K(15, 3)$



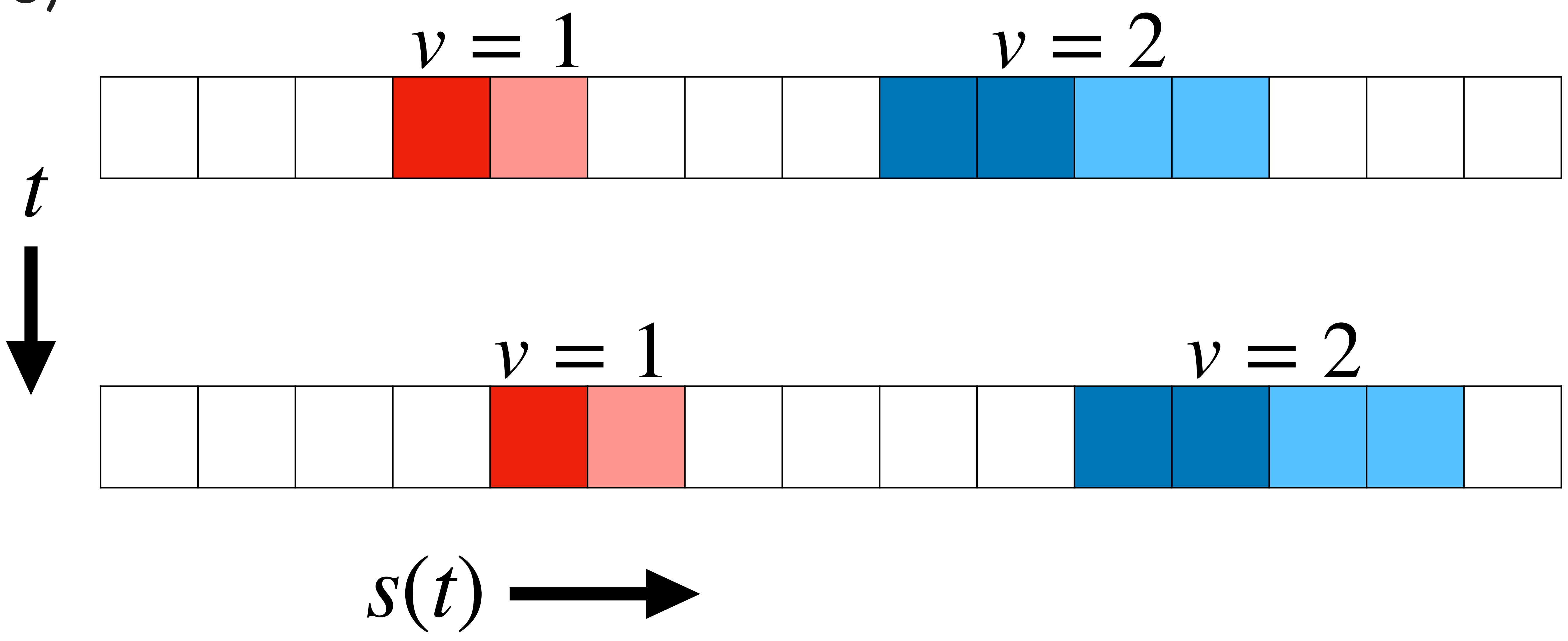
Motion of gliders

$K(15, 3)$



Motion of gliders

$K(15, 3)$



- position after time t : $s(t) = v \cdot t + s(0)$

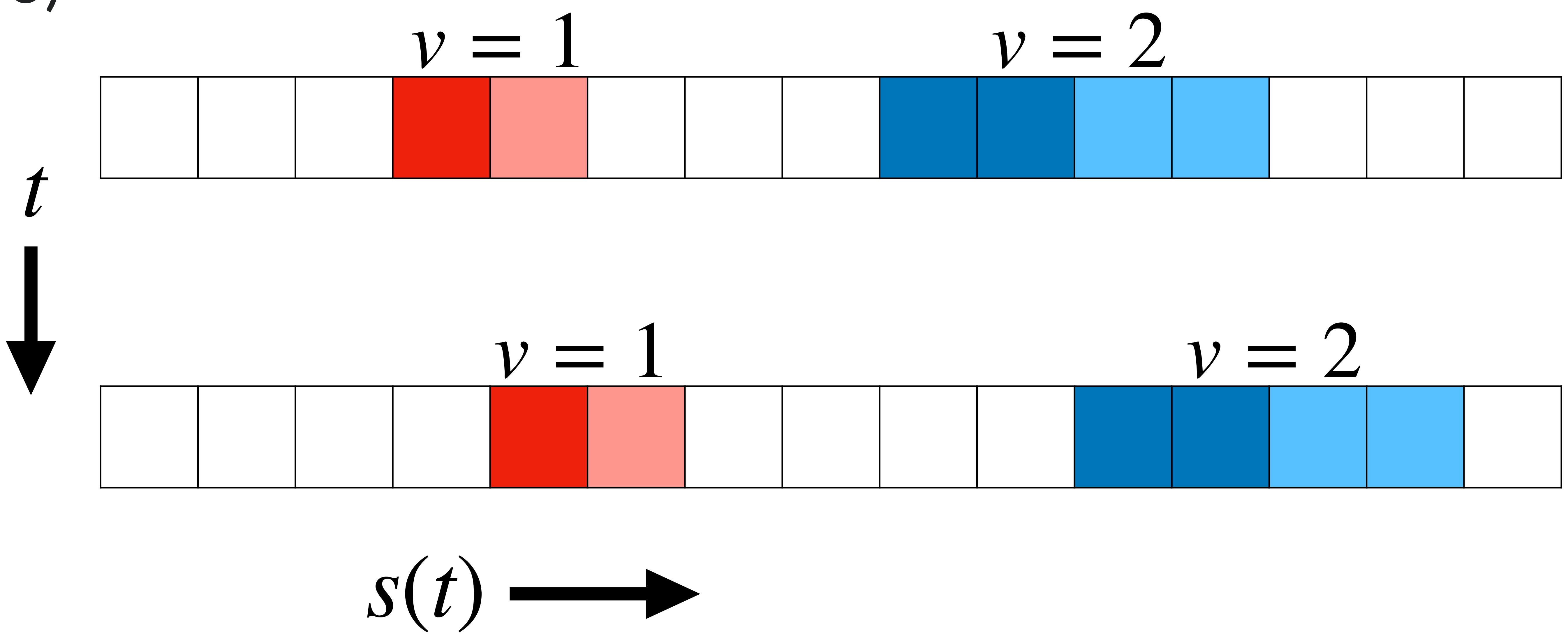
speed

initial position

number of times f is applied

Motion of gliders

$K(15, 3)$



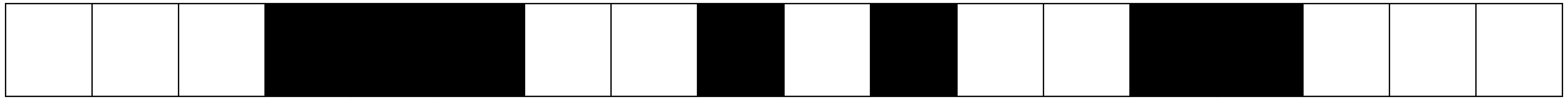
- position after time t : $s(t) = v \cdot t + s(0)$

  speed (v); number of times f is applied (t); initial position ($s(0)$)

- motion can be either uniform or non-uniform

Gliders partitioning

$K(18, 7)$

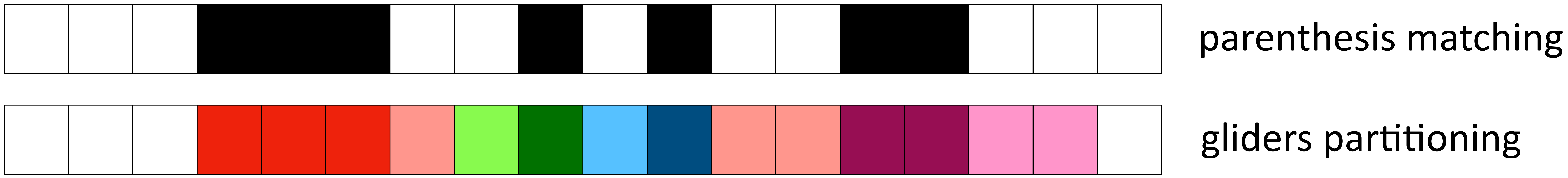


parenthesis matching

Gliders partitioning

$K(18, 7)$

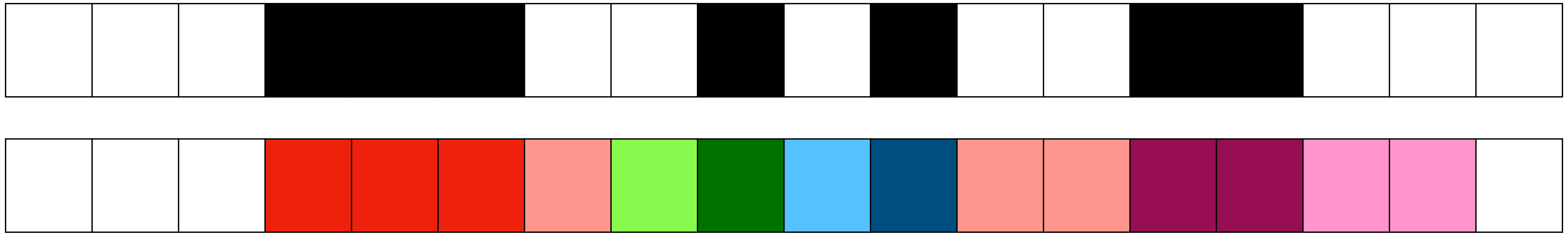
- Assigning matched bits to gliders (coloring boxes) is complicated!



Gliders partitioning

$K(18, 7)$

- Assigning matched bits to gliders (coloring boxes) is complicated!



parenthesis matching

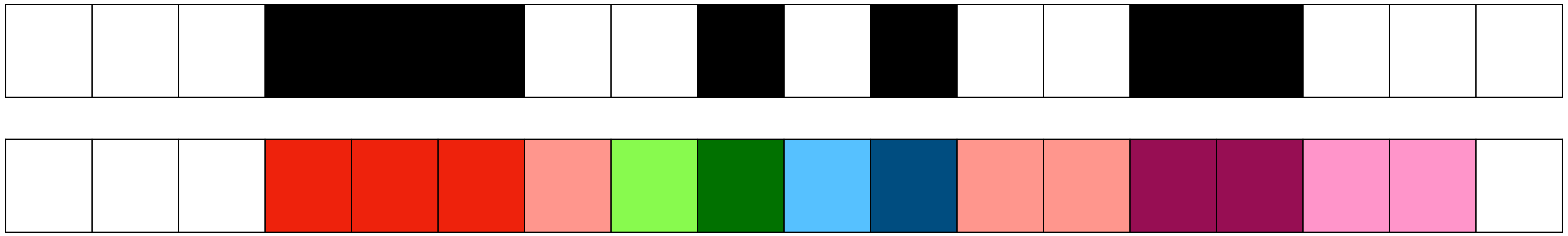
gliders partitioning

- recursion assigns the matched bits to gliders

Gliders partitioning

$K(18, 7)$

- Assigning matched bits to gliders (coloring boxes) is complicated!

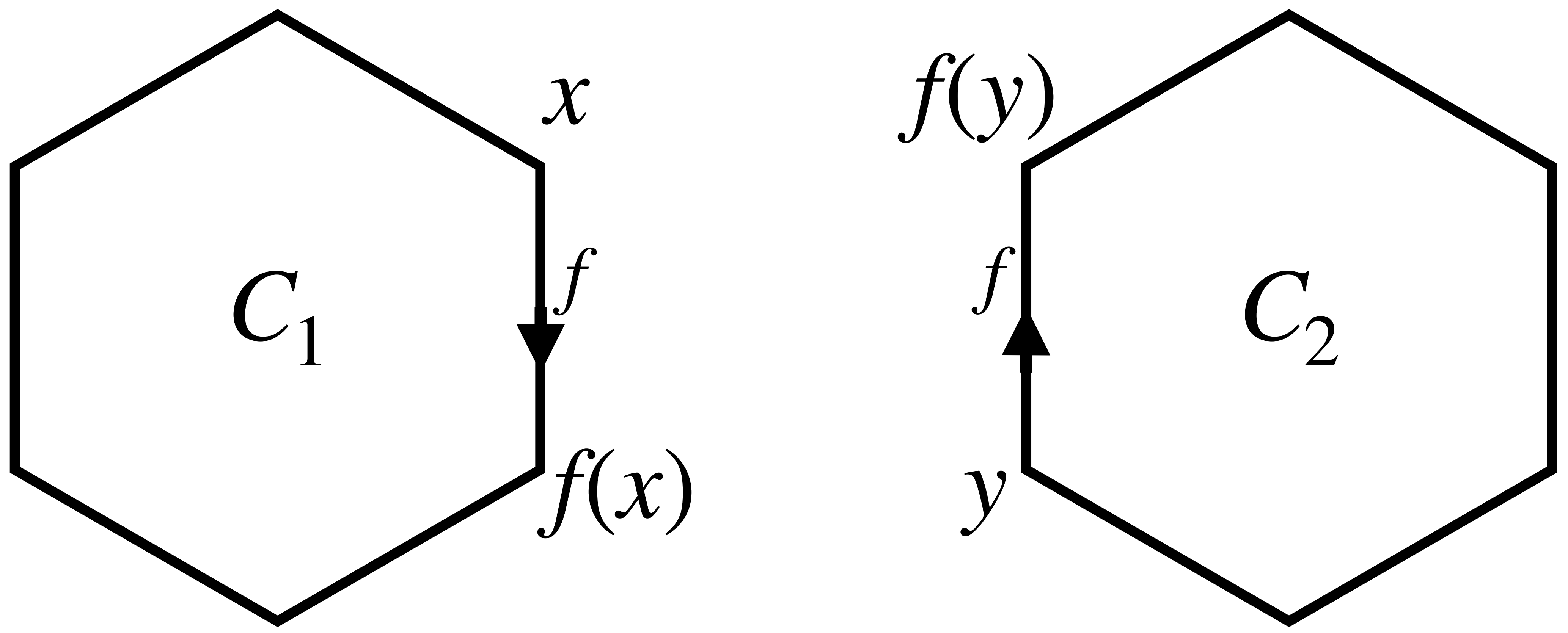


parenthesis matching

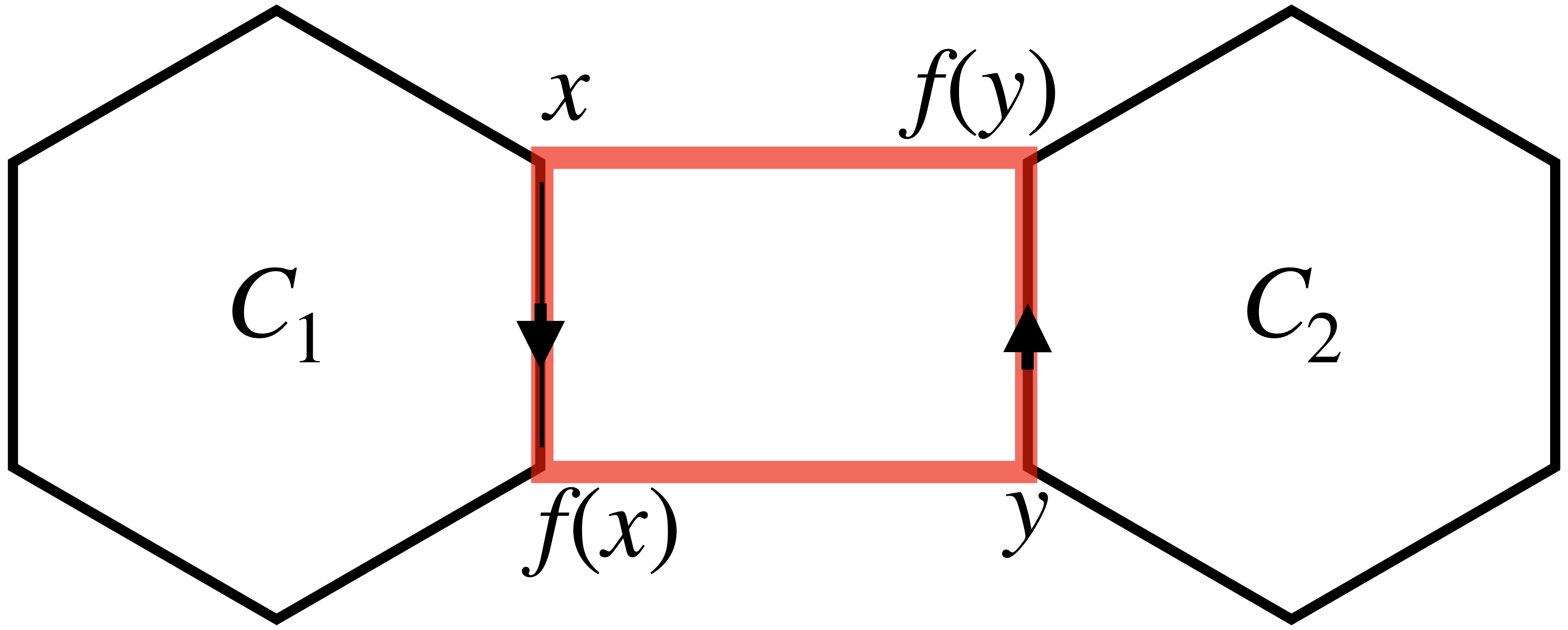
gliders partitioning

- recursion assigns the matched bits to gliders
- speed set of gliders is a **cycle invariant**

Gluing cycles

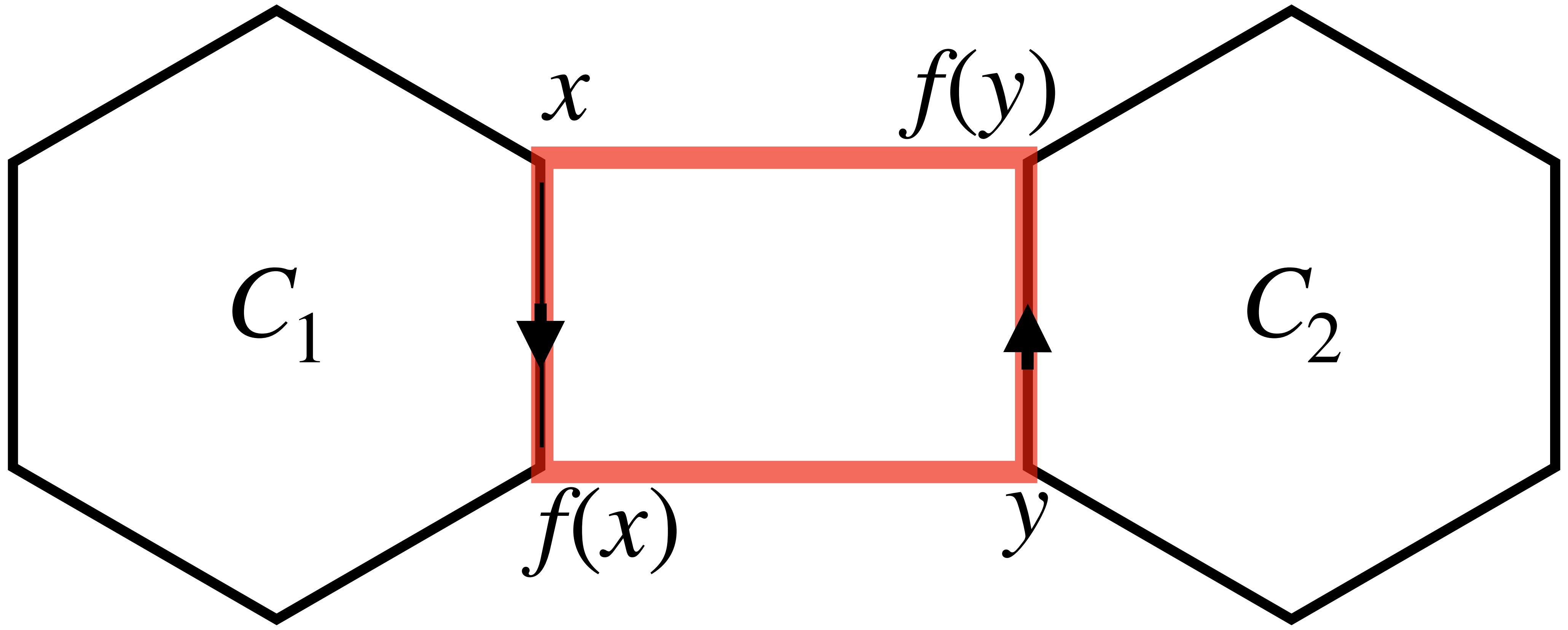


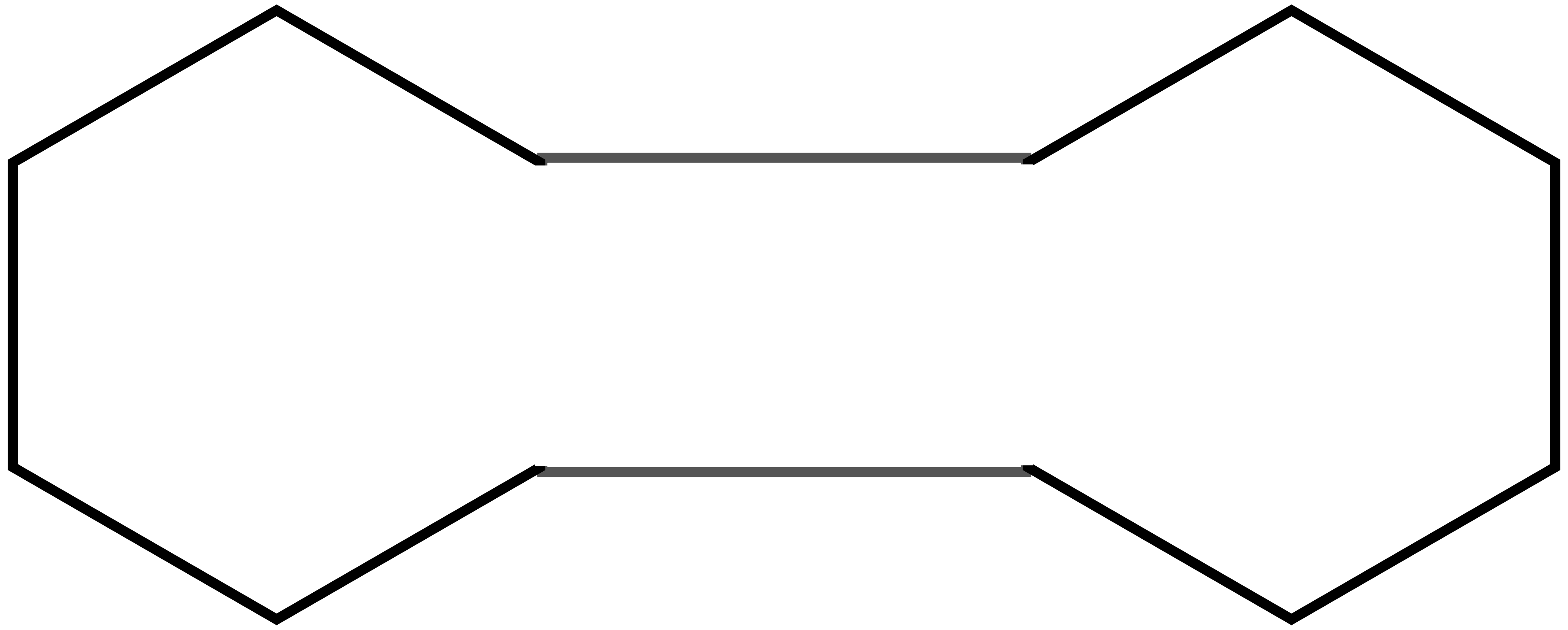
Gluing cycles



- pair (x, y) such that $x\,f(x)\,y\,f(y)$ is a 4-cycle

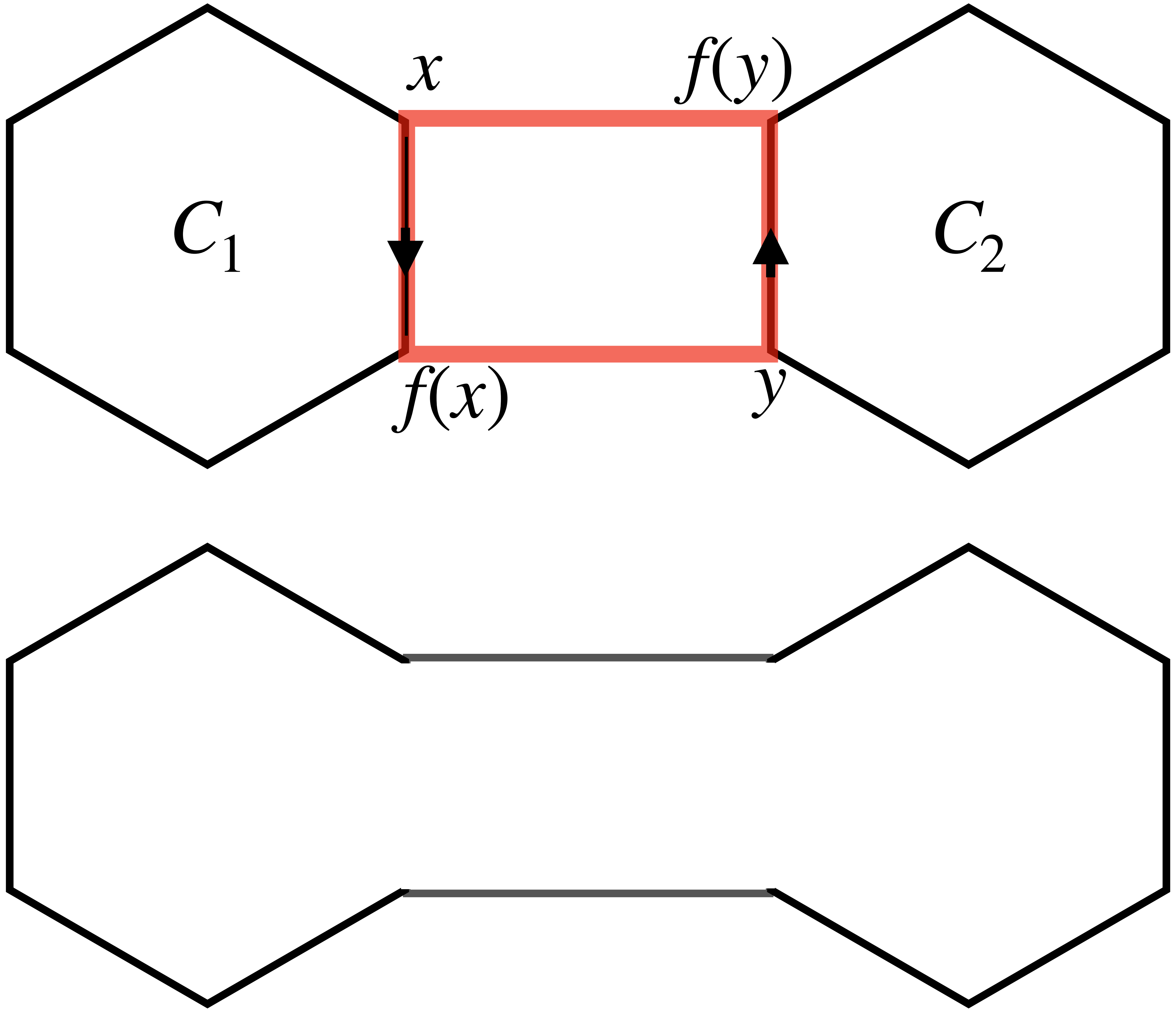
Gluing cycles





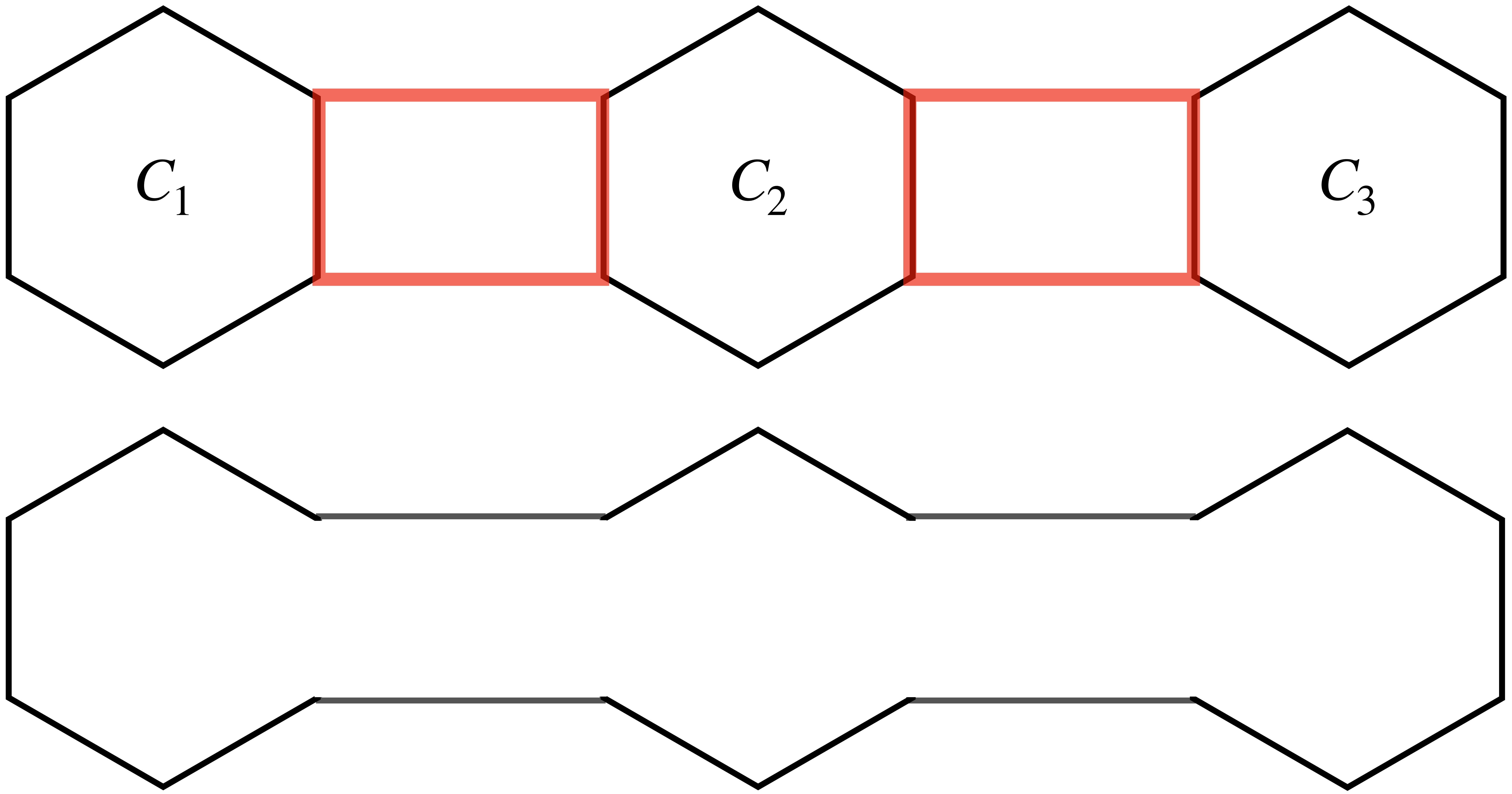
- pair (x, y) such that $x\,f(x)\,y\,f(y)$ is a 4-cycle

Gluing cycles

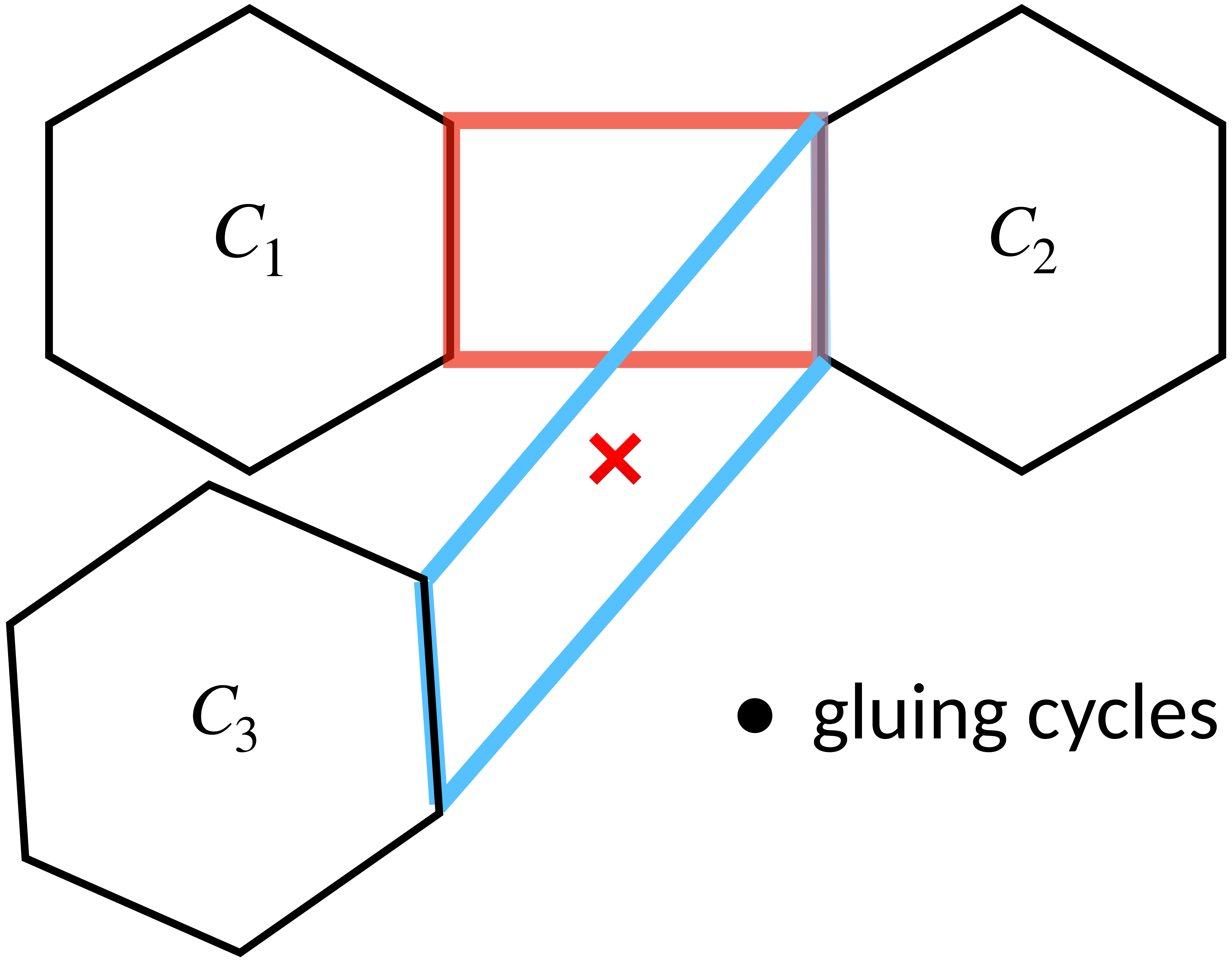


- pair (x, y) such that $x\,f(x)\,y\,f(y)$ is a 4-cycle
  - 4-cycle is there since $n \geq 2k + 3$

Gluing cycles



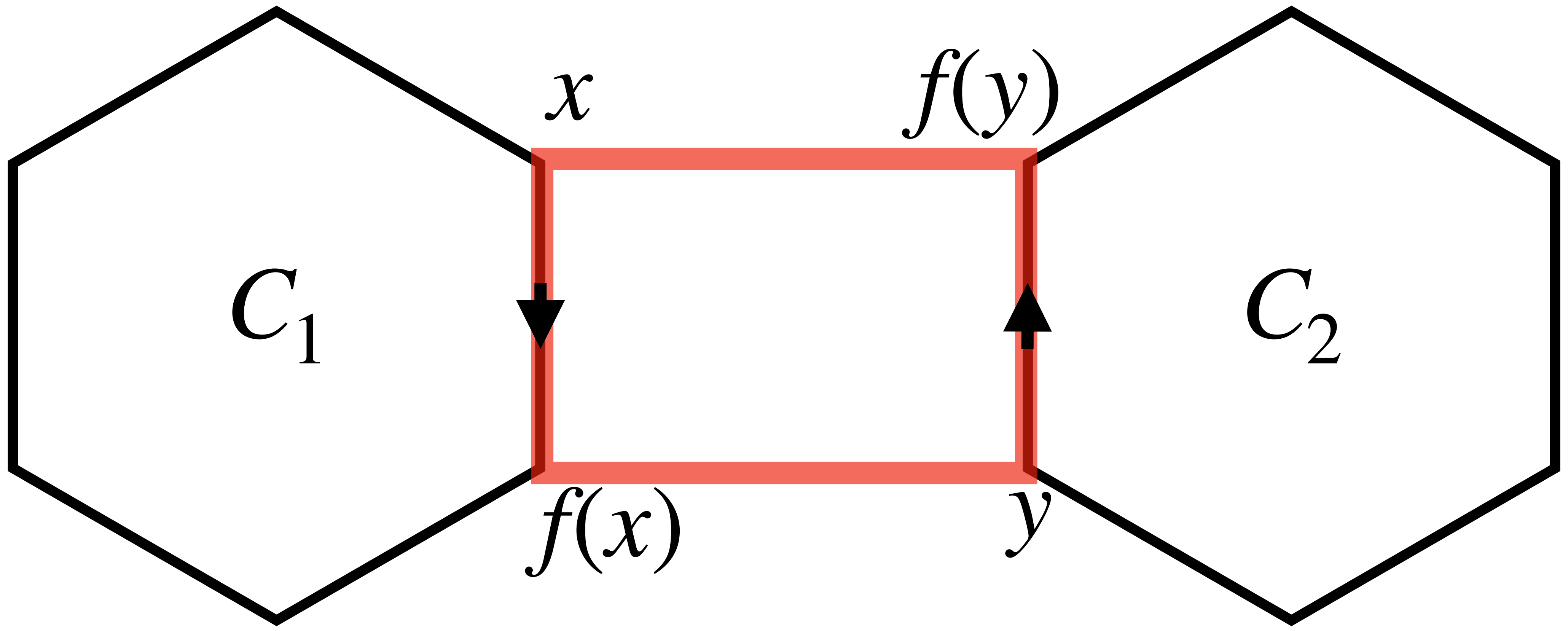
Gluing cycles



- gluing cycles must be edge-disjoint

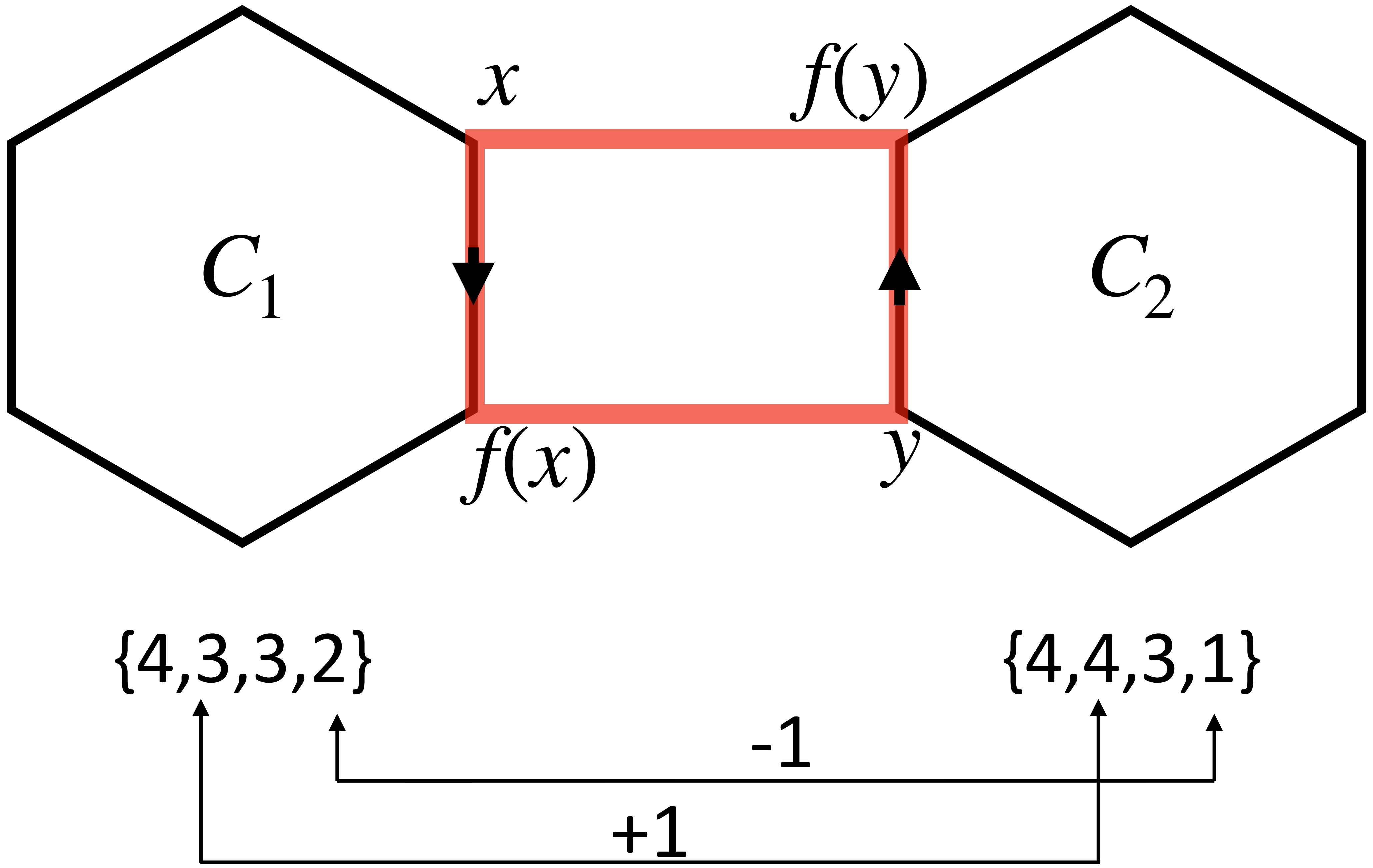
Gluing cycles

- find (x, y) when the speed set of gliders increases lexicographically



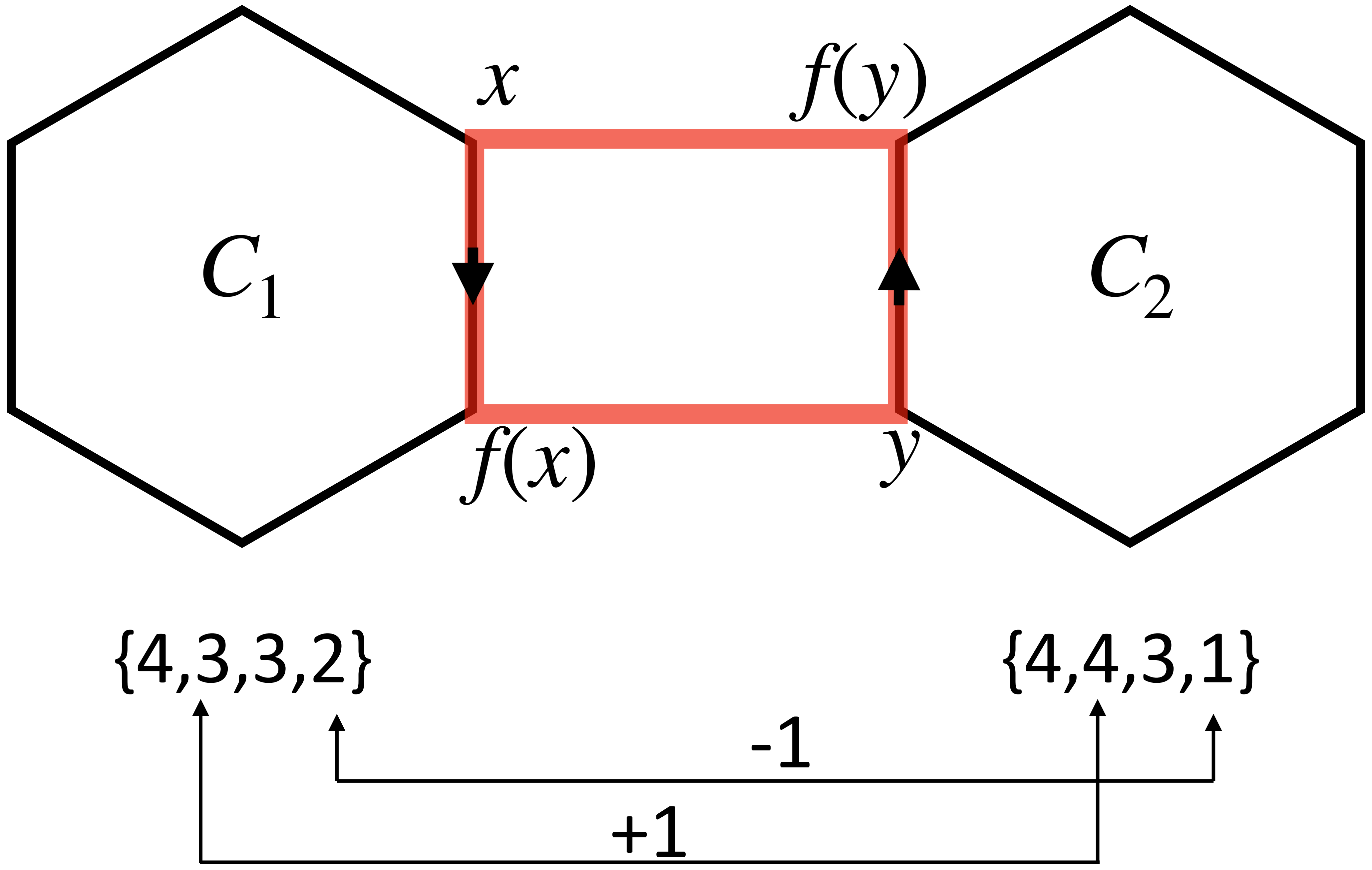
Gluing cycles

- find (x, y) when the speed set of gliders increases lexicographically



Gluing cycles

- find (x, y) when the speed set of gliders increases lexicographically



- ensures connectivity and hence Hamiltonicity

Open questions

- efficient algorithms?

Open questions

- efficient algorithms?
- other vertex transitive graphs?

Open questions

- efficient algorithms?
- other vertex transitive graphs?
- Hamilton decomposition of Kneser graphs?

Thank you!