Multi-layer Cops & Robbers

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Cops and robbers

First the classic version: we play this game on a graph.

- First the cop chooses a vertex
- Then the robber chooses a vertex
- On each subsequent turn:
  - the cop may move along a single edge
  - the robber may move along a single edge

The cop wins if they capture the robber in finite time, the robber otherwise.

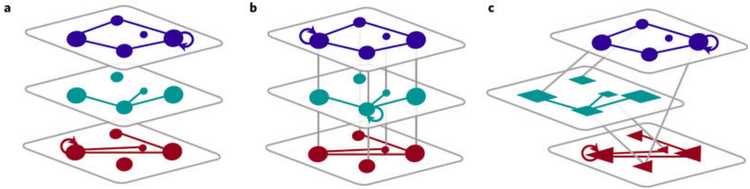
(hopefully we can do a few examples on the visualiser)

Cops and robbers

A few important notions on classic cops and robbers:

- a graph on which a single cop can catch a single robber is called *copwin*
- we can determine if a graph is copwin efficiently using a vertex ordering
- the minimum number of cops required to catch a robber on the graph is the *cop number*, sometimes denoted $c(G)$ for graph G
- dominating sets or dominating paths often play a role in bounds on cop number
- Meyniel's conjecture: all graphs have cop number $O(\sqrt{|V|})$
- for a graph with treewidth w, cop number is at most $\lfloor w/2 \rfloor + 1$

Multi-layer Graphs



(from De Domenico, Granule, Porter, Arenas, 2006. The physics of spreading processes in multi-layer networks, Nature Physics)

Cops & Robbers on Multi-layer Graphs

Here: a multi-layer graph layer$(G_1, \ldots, G_\tau)$ consists of several (not nesc. disjoint) layers $G_1, \ldots, G_\tau$ (think edge sets).

- We restrict each cop to a single layer.

  may be several cops on one layer.

- Robber can use any edge from any layer.

Let mc$(G_1, \ldots, G_\tau)$ be the minimum number of cops that need to be assigned to any layers to catch the robber in layer$(G_1, \ldots, G_\tau)$.

We find lots of slightly counter-intuitive examples.

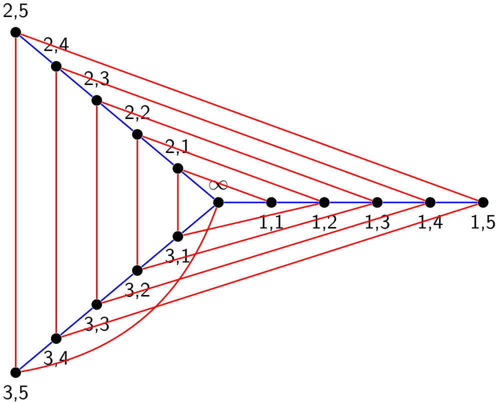
Example: 2D square grid

We divide our grid's edges into two layers - the vertical and the horizontal.

- Then two cops suffice - **but only if they are both in the same layer** (wlog vertical)

- **First** - both cops move to be in the same row as the robber
- **Second** the cop that is initially closer to the robber keeps the robber from entering their column, while the second cop advances until they are in the same row as the robber but one column closer than the other cop

Cop number strictly greater than sum of layers



We have a (mostly awful) argument that two cops are not enough no matter how you layer them.

A little bit of complexity

Given a specification of a graph with its layers the question:

- Is there an assignment of k cops to layers such that the cops can win?

Is **NP-Hard**, even if every cop layer is a tree or a clique

On the other hand, if we limit things such that there are a polynomial number of game states, then resolving the outcome of the game is polytime - similar to Hahn and MacGillivray.

Theorem

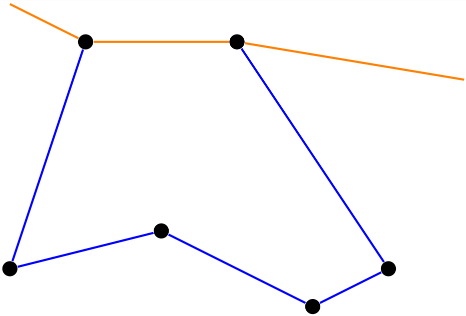
Even if each cop layer is a tree, determining if k cops can win is NP-hard.*

Reduction from dominating set:

- Take a graph, and create a layer for each vertex which includes said vertex and its neighbours.
- Cops must be on a star
- Robber is on original graph
- Cops win if there is a dominating set of size k

Theorem

If the robber layer is a tree, determining if k cops can win is poly-time solvable.



Corollary

If the robber layer is a tree, the cop number can be determined in poly time.

Layers and bounds (not!)

What relationships exist between the cop numbers of the layers and the (multilayer) cop number of the whole system?

If each layer has large cop number does the whole thing have large cop number?

If all layers have small cop number does the whole thing have small cop number?

Any nice functions that give bounds with this flavour?

No.

Relationship between layers and overall - Functional Bound?

Is there a function f such that for any graphs $G_1, \ldots, G_k$

$$\mathrm{mc}(G_1, \ldots, G_\tau) \leq f(\mathrm{c}(G_R), \mathrm{c}(G_1), \ldots, \mathrm{c}(G_\tau)) \text{ ???}$$

NO!

Theorem

For any constant $k > 0$ there exists a multi-layer graph with $c(G_R) \leq 4$, $c(G_1) = c(G_2) = 2$ and $\mathrm{mc}(G_R, G_1, G_2) \geq k$.

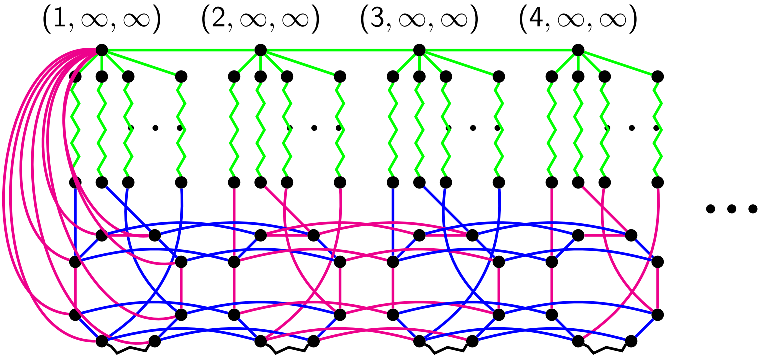


Figure: The whole multi-layer graph. Blue edges are in G_1, pink edges are in G_2, and green edges are in both G_1 and G_2. The green zigzag lines are representations of paths on $5k$ vertices, and the black zigzag lines represent a path on $2k-7$ edges, alternating in colour with both end edges being pink.

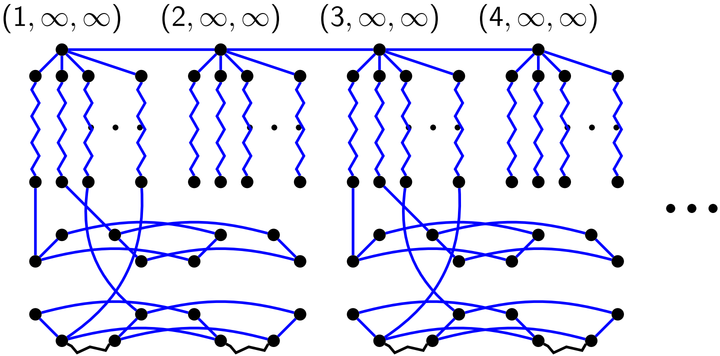


Figure: Only G_1.

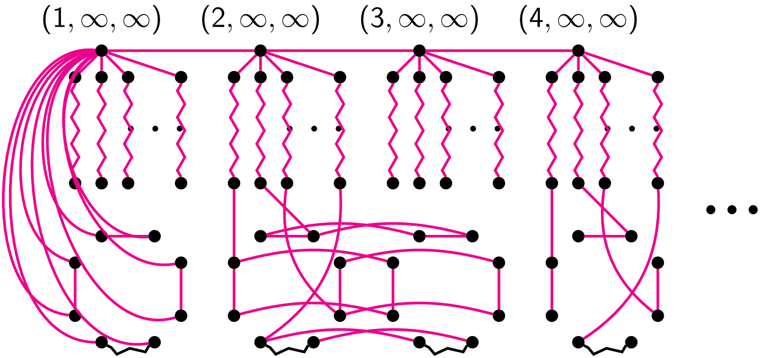


Figure: Only G_2.

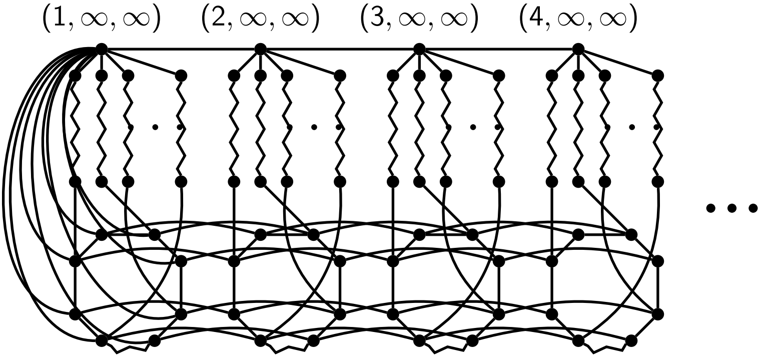


Figure: Only G_R.

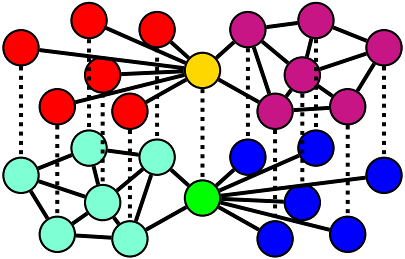
Other Way Round?

Do we have $\mathrm{mc}(G_1, \ldots, G_\tau) \geq \min_{i \in [\tau]} \mathrm{c}(G_i)$???

NO!

Proposition

For any $c \geq 1$ there exist two graphs G_1, G_2 with $\mathrm{c}(G_1), \mathrm{c}(G_2) \geq c$ and $\mathrm{mc}(G_1, G_2) = 2$.



These examples have given us that there aren't nice relationships between cop number in layers and overall in terms of constants - can we do more in terms of cops needed as a growing function?

We'll see an example where number of cops needed is dependent on number of layers

Then an example that gives growing cop number even with few layers and small each-layer cop number (!!!)

Worst Case Multi-layer Cop Number

What is the worst way to divide G into τ layers for the cops?

$$\mathrm{mc}_\tau(G) = \max_{G_1,\ldots,G_\tau \in \mathcal{C}_{|V(G)|}\,:\, G_1 \cup \cdots \cup G_\tau = G} \mathrm{mc}(G_1, \ldots, G_\tau)\,,$$

where $\mathcal{C}_n$ denotes the set of connected graphs on n vertices.

We assume connected spanning layers to avoid pathologies.

Corrupt/stupid police chief creates 'ineffective beats' for honest cops to choose from.

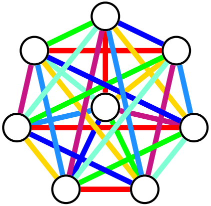
Worst Case Multi-layer Cop Number of the Clique

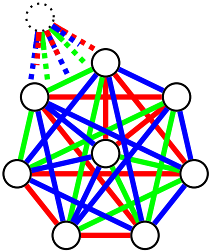
Theorem

Let $n \geq 1$ *and* $1 \leq \tau < \lfloor \frac{n}{2} \rfloor$ *then* $\lceil \frac{\tau}{10} \rceil \leq \mathrm{mc}_\tau(K_n) \leq \tau$.

Upper Bound: Place one cop from each layer at a single vertex v. Every vertex of K_n is adjacent to v in some layer.

Lower Bound: Choosing connected layers in a 'balanced' way based on an edge colouring of the even clique due to Sofier.





Multi-layer Meyniels Conjecture

Meyniel's Conjecture: $\mathcal{O}(\sqrt{n})$ cops suffice on any graph G.
Is there a Multi-layer analogue of Meyniels Conjecture?

The clique shows that τ many cops are necessary, in general.

For any fixed τ and any connected n-vertex graphs $G_1, \ldots, G_\tau$, $\mathrm{mc}(G_1, \ldots, G_\tau)$ – what bounds can we find?

Example: 3 Layers but Massive Cop Number

Theorem

The exist three n-vertex graphs G_1, G_2, G_3 each with cop number 2 such that if $G_R = G_1 \cup G_2 \cup G_3$ then $\mathrm{mc}(G_R, G_1, G_2, G_3) = \Omega\left(\frac{n}{\log n}\right)$.

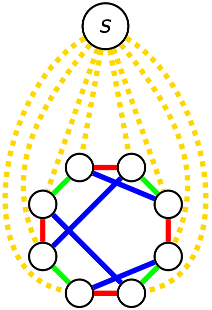
Construction:

-Take a 3-regular 3 edge-colourable α-expander G.

-Each color class in G is a cop layer.

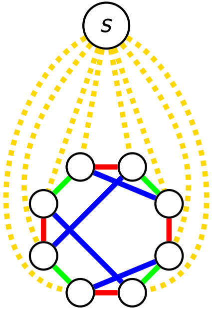
-Add paths of length $3D = \Theta(\log n)$ from each vertex in $V(G)$ to a new vertex s (these edges are in all cop layers).

- Robber can use any edge.



Theorem (Bagchi et al. 2006)

Let G be any 3-regular α-expander G and $S \subset V(G)$ where $|S| \leq \alpha|V(G)|/48$. Then there exists an $\alpha/2$-expander $H \subseteq G[V(G) \setminus S]$ with $|V(H)| \geq 2n/3$.



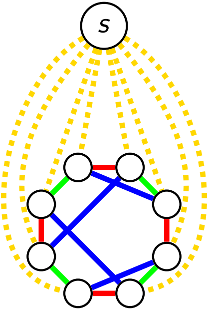
Why Does This Work?

Theorem

We can find lots of expanders in our expander

Idea:

- \- Choose $D = C \log |V(G)|$ longer than the max diameter of any H above.
- \- A cop can only visit at most 2 distinct vertices if $V(G)$ within any $6D - 1$ steps.
- \- Robber (lives on G) & finds a $H \subseteq G$ that is 'safe' from any cop at distance at most $2D + 1$ from $V(G)$.
- \- Robber tries to stay in such a 'safe' H at each step. If cops threaten H, robber has time to find a new H' and move (safely) in H to H' before cops arrive.



Multi-layer Cop Number Vs. Treewidth

Proposition (Joret, Kamiński & Theis 2010)

The cop number of a graph G is at most $\mathrm{tw}(G)/2 + 1$.

By a similar strategy (move to the robber one bag at a time):

Proposition

For any connected graphs $G_1, \ldots, G_\tau$ on a common vertex set,

$$\mathrm{mc}(G_1, \ldots, G_\tau) \leq \mathrm{tw}(G_1 \cup \cdots \cup G_\tau).$$

Furthermore, the multi-layer graph $\mathrm{layer}(G_1, \ldots, G_\tau)$ *can be guarded by* $\mathrm{tw}(G_1 \cup \cdots \cup G_\tau)$ *cops placed in any layers.*

Wrap-up and Thanks!

Thanks for your attention through pictures and technical!

- Multi-layer cops and robbers has surprised us so far
  - Various examples showing that there is no guaranteed relationship between cop numbers of layers and of whole system
  - we can generate graphs that require $\Omega\left(\frac{n}{\log n}\right)$ cops
  - Mostly negative results - but we think interesting!

Questions? I'll do my best!

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