Laguerre digraphs and continued fractions

Bishal Deb (he/him)

University College London

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Based on Joint Work With

Alex Dyachenko, Matthias Pétréolle, Alan Sokal

Structure

1. Laguerre digraphs
2. Combinatorics of continued fractions
3. Jacobi–Rogers matrix
4. Biane history
5. Foata–Zeilberger history
6. List of applications

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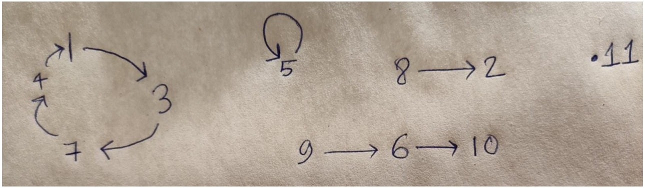
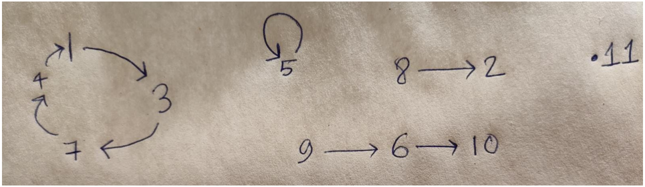
Example:



Connected components



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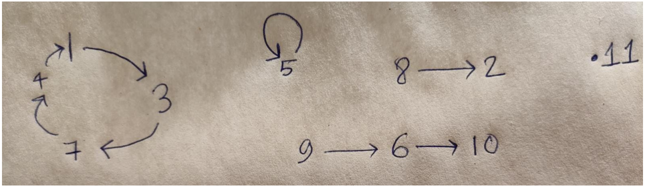

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- Directed cycle
- Directed paths

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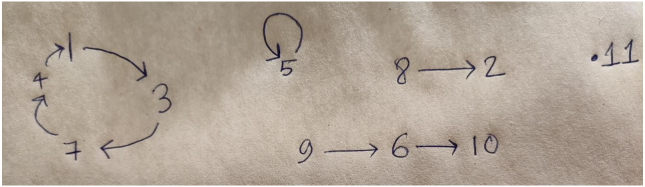

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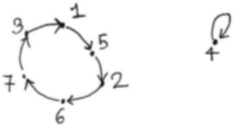

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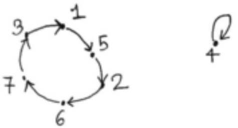

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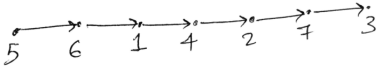
2. One path, no cycles - linear structure of permutation



$\sigma = 5614273$

Enumeration

$\mathrm{LD}_{n,k}$ - Set of Laguerre digraphs on n vertices with k paths

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$$\sum_{n=0}^{\infty} \sum_{G \in \mathrm{LD}_n} \lambda^{\mathrm{cyc}(G)} x^{\mathrm{pa}(G)} \frac{t^n}{n!} = \exp\left(\frac{xt}{1-t} + \lambda \log \frac{1}{1-t} \right)$$

In particular, $\mathrm{LD}_{n,k}$ *is enumerated by*

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Laguerre polynomials

Laguerre polynomials are a sequence of orthogonal polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

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$$\mathcal{L}_n^{(\alpha)}(x) = n! L_n^{(\alpha)}(-x) = \sum_{k=0}^{n} \binom{n}{k} (n+\alpha)(n-1+\alpha)\cdots(k+1+\alpha)x^k$$

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Laguerre digraphs after Sokal (2022)

Structure

Combinatorial Interpretation of J-fraction

Jacobi-type continued fraction (J-fraction)

$$\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\gamma_2 t-\cfrac{\beta_3 t^2}{\ddots}}}}$$

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Associated C-fraction outside of combinatorial literature

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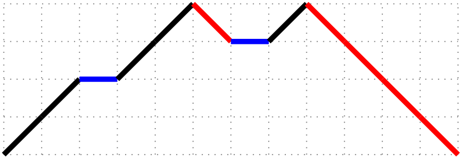

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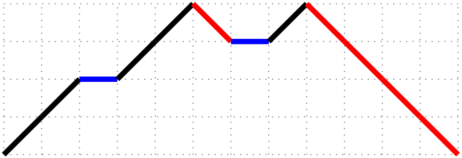

Assign weights:

- $\nearrow\ :\ 1$
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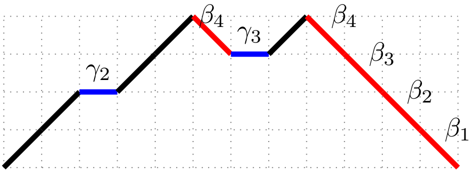

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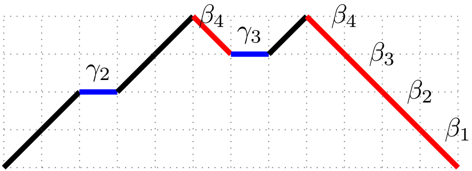

Weight $= \beta_1 \beta_2 \beta_3 \beta_4^2 \gamma_2 \gamma_3$

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The a_n are weighted sum of Motzkin paths with n steps.

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Gateway for proving continued fractions using bijective combinatorics :-D

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Jacobi–Rogers Matrix

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Construct matrix J with entries

$$\mathrm{J}_{n,k} \;=\; \text{Weighted sum of partial Motzkin paths } (0,0) \text{ to } (n,k)$$

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Also known as Stieltjes table/tableau

If

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Question: If J-fraction for a_n is known, combinatorially understand matrix J

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5. Foata–Zeilberger history
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Jacobi-type continued fraction for $n!$:

$$1 + 1!t + 2!t^2 + 3!t^3 + 4!t^4 + \ldots = \cfrac{1}{1 - 1 \cdot t - \cfrac{1 \cdot t^2}{1 - 3 \cdot t - \cfrac{4 \cdot t^2}{1 - 5 \cdot t - \cfrac{9 \cdot t^2}{1 - \ddots}}}}$$

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Each permutation σ corresponds to (ω, ξ) where ω is Motzkin path and choice of labels ξ

Construction of path for $n!$

In the Foata–Zeilberger and Biane bijections path is the same labels are different
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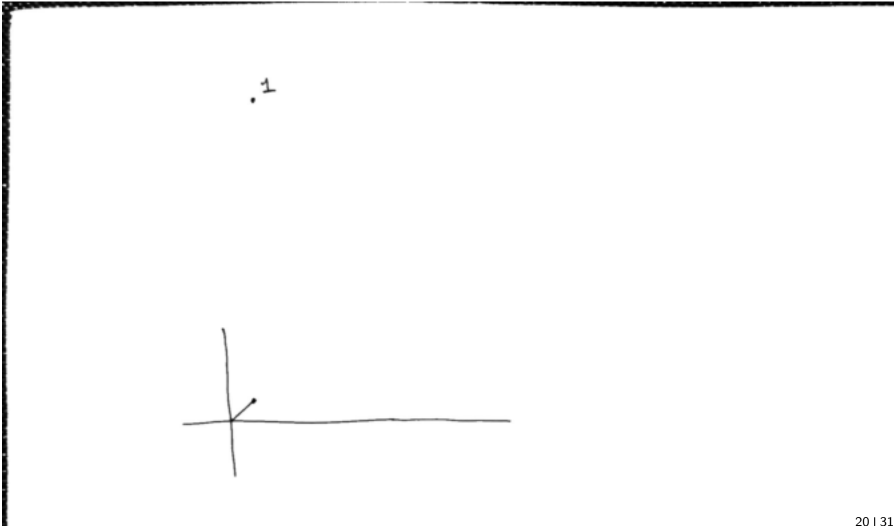

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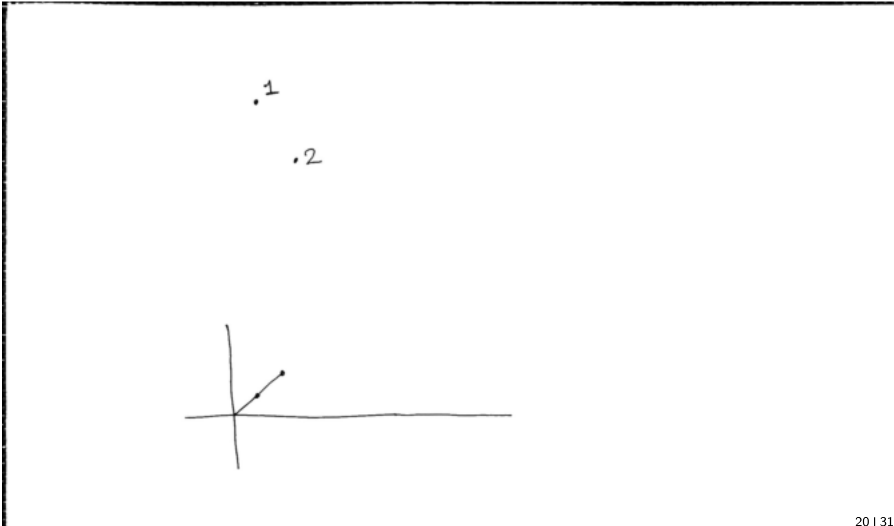

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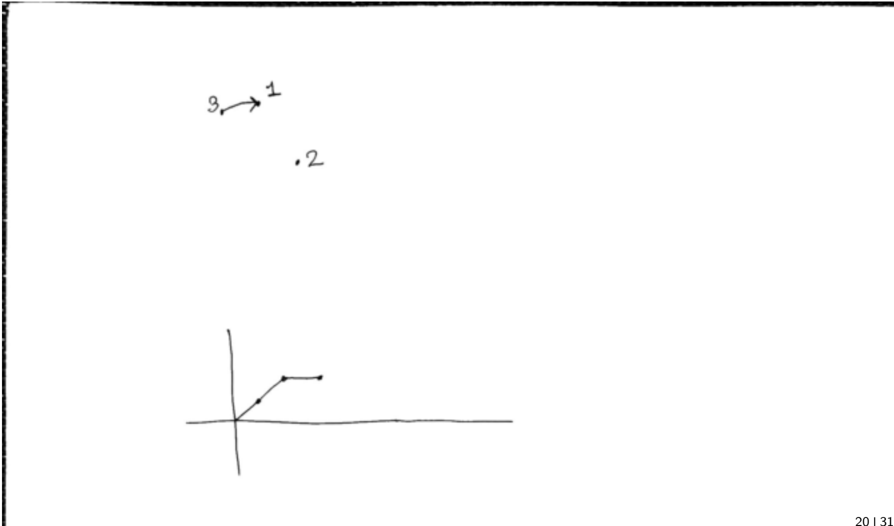

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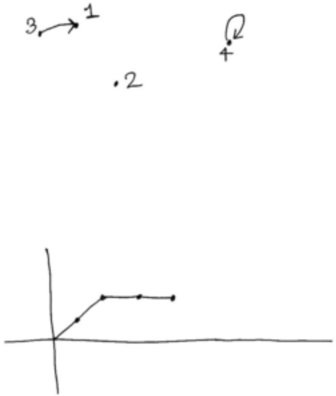

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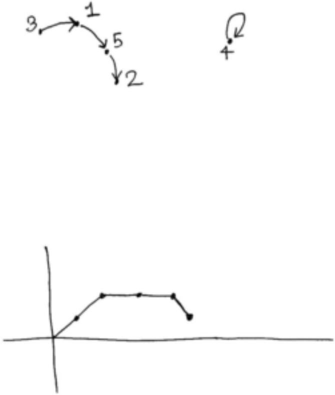

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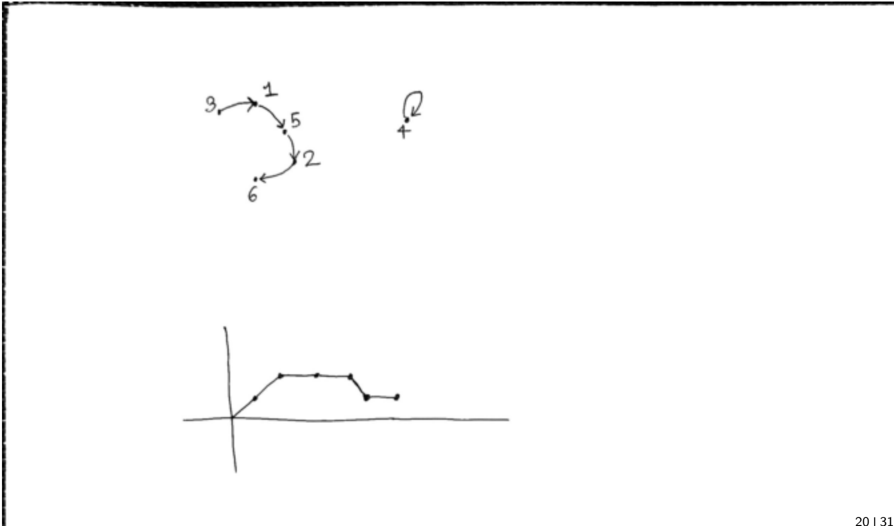

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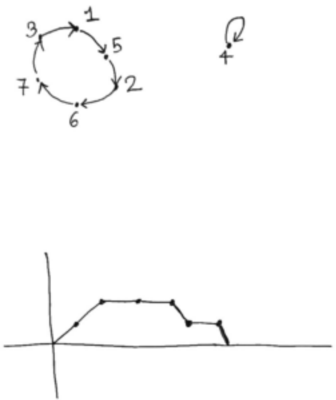

When $a_n = n!$,

$$\sum_{n=0}^{\infty} a_n t_n = \cfrac{1}{1 - t - \cfrac{1t^2}{1 - 3t - \cfrac{4t^2}{1 - \ddots}}}$$

$$\mathrm{J}_{n,k} = \binom{n}{k}\frac{n!}{k!}$$

These count Laguerre digraphs with k paths

Biane history

Flag of Laguerre digraphs exhibiting Biane's construction

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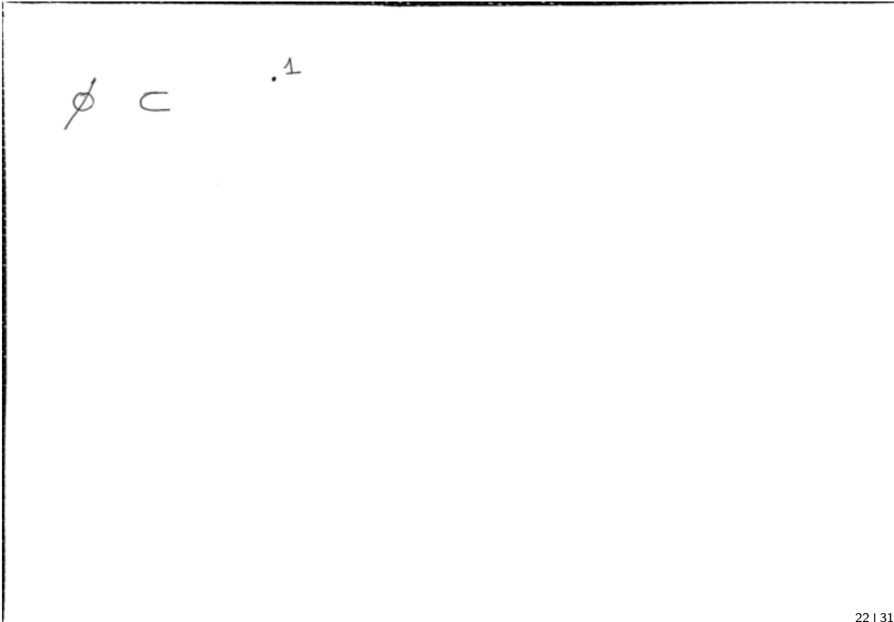
$\emptyset \subset$

Flag of Laguerre digraphs exhibiting Biane's construction



Biane history

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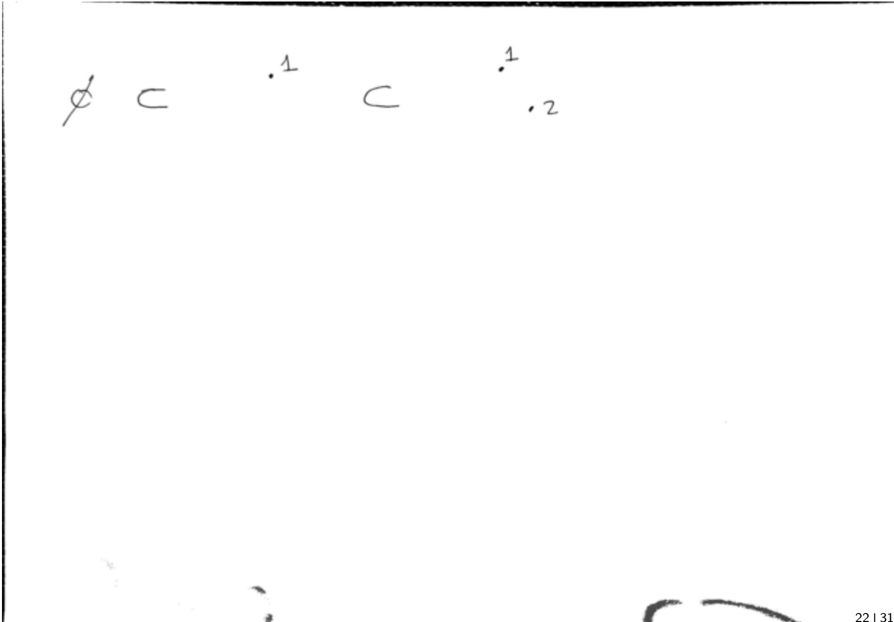

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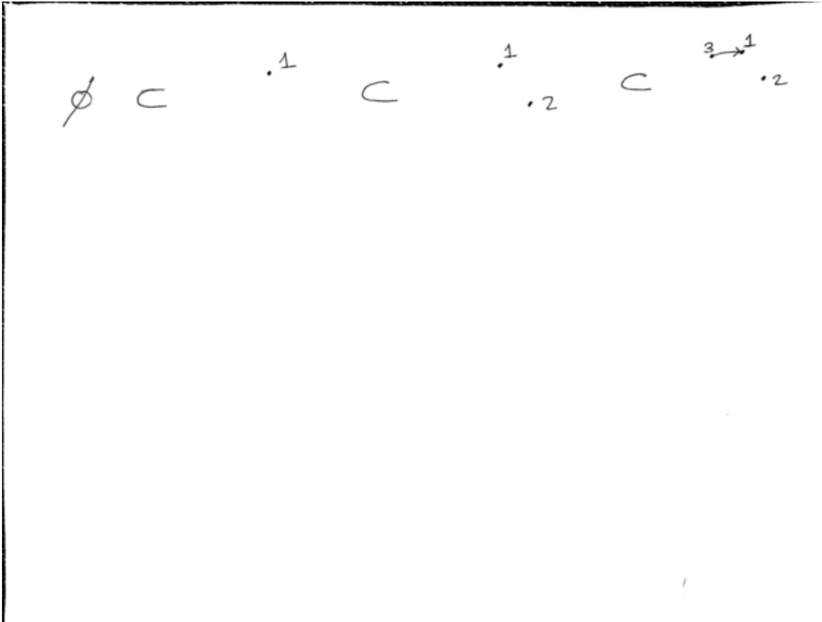

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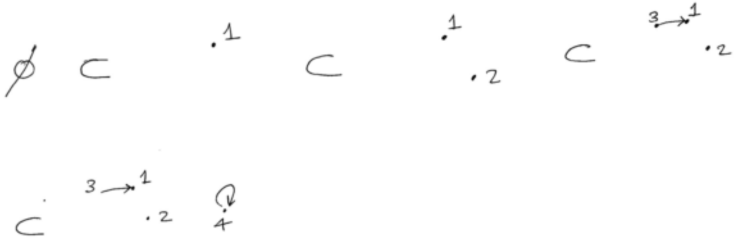

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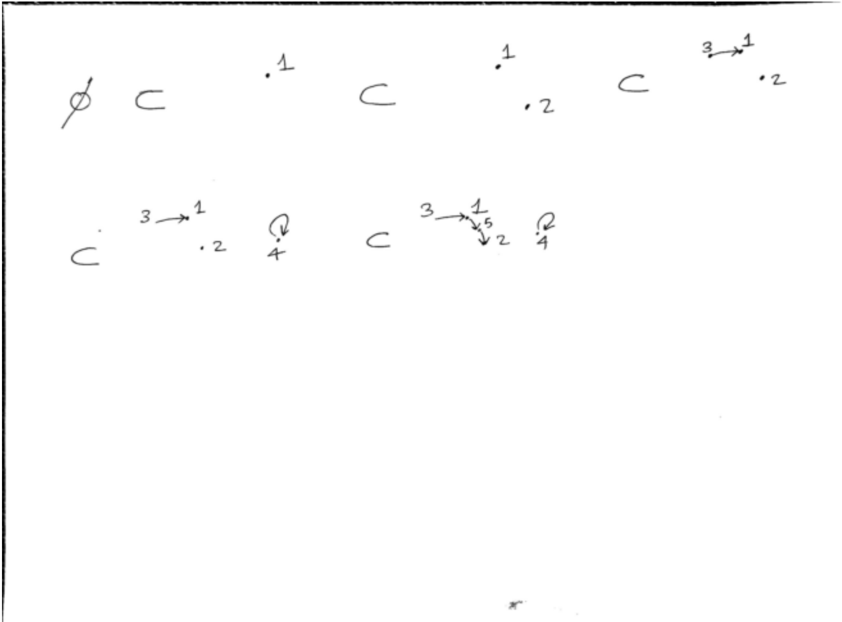

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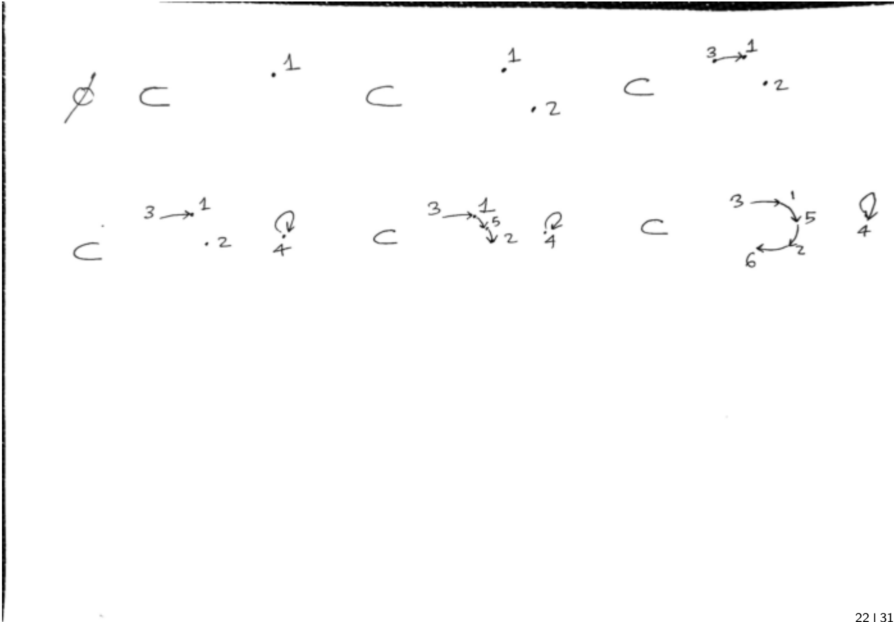

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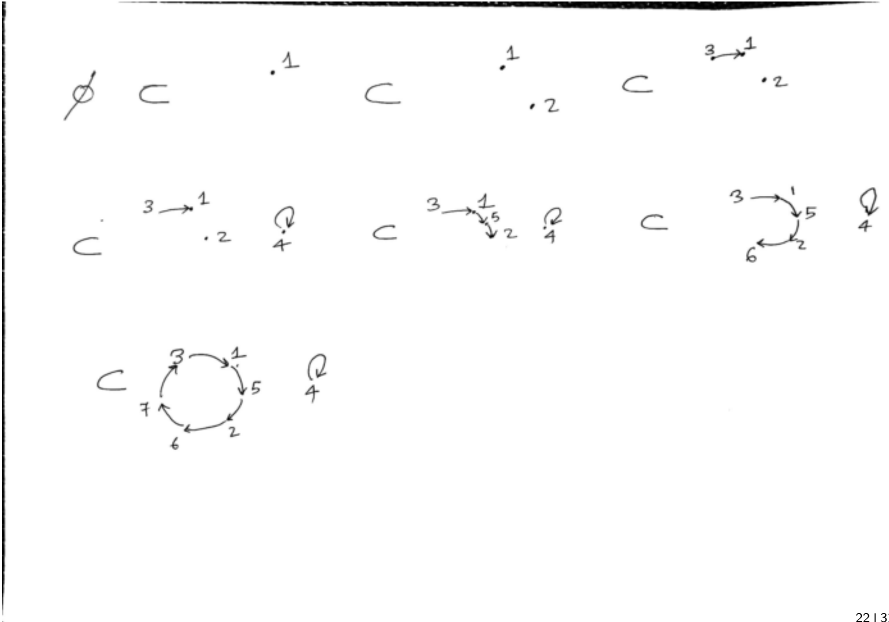

Structure

Foata–Zeilberger history

Insertion of edges rather than vertices at each step

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Define sets

- excedance indices $F = \{i \in [n] : \sigma(i) > i\}$
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- fixed points H

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Twist in story: Can keep track of cycles being created using Foata–Zeilberger bijection

Structure

1. Laguerre digraphs
2. Combinatorics of continued fractions
3. Jacobi–Rogers matrix
4. Biane history
5. Foata–Zeilberger history
6. List of applications

Cycle classification

For a permutation σ, compare each i with $\sigma(i)$ and $\sigma^{-1}(i)$:

Cycle classification

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- cycle valley $\sigma^{-1}(i) > i < \sigma(i)$
- cycle peaks $\sigma^{-1}(i) < i > \sigma(i)$
- cycle double rise $\sigma^{-1}(i) < i < \sigma(i)$
- cycle double fall $\sigma^{-1}(i) > i > \sigma(i)$
- fixed point $i = \sigma(i) = \sigma^{-1}(i)$

Record classification

Consider σ as a word $\sigma(1)\sigma(2)\dots\sigma(n)$:

- i is record if for every $j < i$ we have $\sigma(j) < \sigma(i)$
  left-to-right-maxima
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Each i is one of the following four types:

- rar - record-antirecord 
- erec - exclusive record 
- earec - exclusive antirecord 
- nrar - neither record-antirecord 

Record-and-cycle classification

Each i is one of the following ten (not 20) types:

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- rar
- nrfix

Continued fractions counting permutation statistics

Consider 11-variable polynomials

$$P_n(x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2, w, z) = \sum_{\sigma \in \mathfrak{S}_n} x_1^{\mathrm{eareccpeak}(\sigma)} x_2^{\mathrm{eareccdfall}(\sigma)} y_1^{\mathrm{ereccval}(\sigma)} y_2^{\mathrm{ereccdrise}(\sigma)} z^{\mathrm{rar}(\sigma)} \times u_1^{\mathrm{nrcpeak}(\sigma)} u_2^{\mathrm{nrcdfall}(\sigma)} v_1^{\mathrm{nrcval}(\sigma)} v_2^{\mathrm{nrcdrise}(\sigma)} w^{\mathrm{nrfix}(\sigma)} \lambda^{\mathrm{cyc}(\sigma)}$$

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Theorem (D. (2023), Conjectured by Sokal–Zeng (2022))

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, y_1, y_2, u_1, u_2, y_1, v_2, w, z, \lambda)t^n$$

$$= \cfrac{1}{1 - \lambda z \cdot t - \cfrac{\lambda x_1 y_1 \cdot t^2}{1 - (x_2 + y_2 + \lambda w)\cdot t - \cfrac{(\lambda+1)(x_1+u_1)y_1 \cdot t^2}{1 - ((x_2+v_2)+(y_2+v_2)+\lambda w)\cdot t - \cfrac{(\lambda+2)(x_1+2u_1)y_1\cdot t^2}{1-\ddots}}}}$$

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Can also prove a 4-variable continued fraction conjectured in 1996 by Randrianarivony–Zeng.

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  Branched continued fractions for Laguerre polynomials and total positivity

Thank you