Conjugacy, languages and groups

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Plan for today

1 Introduction

2 Formal language theory
   Machines
   Series

3 Groups
   Building languages from groups
   Right-angled Artin groups

4 Results
Aim of today: find out which 'conjugacy language' properties are preserved under quasi-isometries.
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Formal languages

Notation:

- \( X = \) finite set.
- \( X^* = \) set of all finite words over \( X \).

Example 1: Let \( X = \{0, 1\} \). Then \( X^* \) is the set of all binary words.
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Example 2

\( X = \{a, b, c, \ldots, z\} \).
\( L = \) English Language (Collins Dictionary 2023).
Chomsky hierarchy

Regular ⇔ Finite-state automaton.
Context-free ⇔ Pushdown automaton.

Figure: Chomsky hierarchy
Chomsky hierarchy

Regular ⇔ Finite-state automaton.
Context-free ⇔ Pushdown automaton.
Finite-state automaton

Figure: Finite-state automaton (FSA)
$\begin{align*}
X &= \{a, b\}, \\
L &= \text{all words over } X \text{ which contain an even number of } b\text{'s.}
\end{align*}$
$L =$ language. Define $\phi_L(n) = |\{w \in L \mid \ell(w) = n\}|.$
Series

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**Definition**

The strict growth series $f_L(z)$ is defined as the infinite series:

$$f_L(z) := \sum_{i=0}^{\infty} \phi_L(i) z^i$$

- $L$ is regular $\Rightarrow f_L(z)$ is rational.
- $L$ is unambiguous context-free (UCF) $\Rightarrow f_L(z)$ is algebraic (Chomsky-Schützenberger).
- If $f_L(z)$ is transcendental $\Rightarrow L$ is not UCF (therefore not regular).
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$G = \langle X \rangle$ finitely generated group. $\pi : X^* \rightarrow G$. For $g \in G$, define:

$|g| : = \text{length of shortest representative word for } g \text{ over } X$.

$[g]_{c} : = \text{conjugacy class of } g$.

$|g|_{c} : = \min \{ |h| : h \in [g]_{c} \} = \text{length up to conjugacy}$.

Conjugacy geodesic language

$\text{ConjGeo}(G, X) : = \{ w \in X^* | \ell(w) = |\pi(w)|_{c} \}$

'Words which are shortest with respect to their conjugacy class'.
Languages from groups

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Example: \( G = F_2 = \langle a, b \rangle \).
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\( \text{ConjGeo}(F_2, X) \) is precisely the set of all cyclically reduced words:

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e.g. \( w = a^{-1}bba \). Consider conjugating \( w \) by \( a \):

\[ awa^{-1} = aa^{-1}bbaa^{-1} = bb. \]
For each conjugacy class $c$, let $z_c$ be the shortlex least word over $X$ representing an element of $c$.

**Example**

Let $X = \{a, b\}$. Order generators as $a < a^{-1} < b < b^{-1}$.
Then for example:

$$aab^{-1}a < ab^{-1}aa$$
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**Shortlex conjugacy language**

$$\text{ConjSL}(G, X) := \{z_c \mid c \in G/ \sim\}$$

‘Unique representative from each conjugacy class’.
Shortlex conjugacy language

\[ \text{ConjSL}(G, X) := \{ z_c \mid c \in G/\sim \} \]

If \( L = \text{ConjSL}(G, X) \), \( f_L(z) = \text{conjugacy growth series} \).
Shortlex conjugacy language

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RECAP:

- \( L \) is regular \( \Rightarrow \) \( f_L(z) \) is rational.
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Ok, time for some groups!
Definition

Let \( \Gamma \) be a finite simple graph. The **right-angled Artin group** on \( \Gamma \), denoted \( A_{\Gamma} \), is the group defined by the following presentation:

\[
A_{\Gamma} = \langle V(\Gamma) \mid [u, v] = 1 \text{ if and only if } \{u, v\} \in E(\Gamma) \rangle
\]
My favourite groups!

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Summary

RAAGs

\[ A_\Gamma \leq G ? \]

Languages

ConjGeo

ConjSL
### Results

<table>
<thead>
<tr>
<th>Group</th>
<th>ConjGeo</th>
<th>ConjSL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Word hyperbolic</td>
<td>Regular*[1]</td>
<td>Not UCF [2]</td>
</tr>
<tr>
<td>XL - type Artin</td>
<td>Regular [1]</td>
<td>Not regular [1]</td>
</tr>
<tr>
<td>Virtually abelian</td>
<td>PT [1]</td>
<td>-</td>
</tr>
<tr>
<td>Virtually cyclic</td>
<td>Regular*</td>
<td>Regular*[1]</td>
</tr>
<tr>
<td>RAAGs</td>
<td>Regular [3]</td>
<td>Not CF [C, 22]</td>
</tr>
<tr>
<td>(Some) virtual RAAGs</td>
<td>...</td>
<td>Not CF [C, 22]</td>
</tr>
</tbody>
</table>

*: true for all generating sets.
PT: Piecewise testable.
Theorem (C, 2022)

Let $A_\phi$ be a virtual RAAG of the form $A_\phi = A_{\Gamma} \times \mathbb{Z}/m\mathbb{Z}$. Then there exists examples of RAAGs $A_{\Gamma}$ such that:

1. $\text{ConjGeo}(A_\phi, X)$ is regular, or
2. $\text{ConjGeo}(A_\phi, X)$ is not context-free.
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1. \( \text{ConjGeo}(A_\phi, X) \) is regular, or
2. \( \text{ConjGeo}(A_\phi, X) \) is not context-free.

Key fact: quasi-isometries don’t guarantee same types of languages!
Yago Antolín, Laura Ciobanu (2017)
Formal Conjugacy Growth in Acylindrically Hyperbolic groups

Laura Ciobanu, Susan Hermiller (2013)
Conjugacy growth series and languages in groups
*Transactions of the American Mathematical Society* 366(5):2803-2825

Laura Ciobanu, Susan Hermiller, Derek Holt, Sarah Rees (2016)
Conjugacy languages in groups
*Israel Journal of Mathematics* 211(1):311-347

Gemma Crowe (2022)
Conjugacy languages in virtual graph products
https://arxiv.org/abs/2212.07111
Thank you for listening!

Any questions?