Conjugacy, languages and groups

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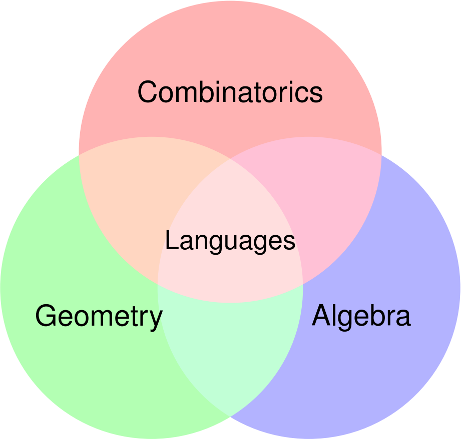
Scottish Combinatorics Meeting
University of Strathclyde
22nd-23rd May 2023

Plan for today

Introduction



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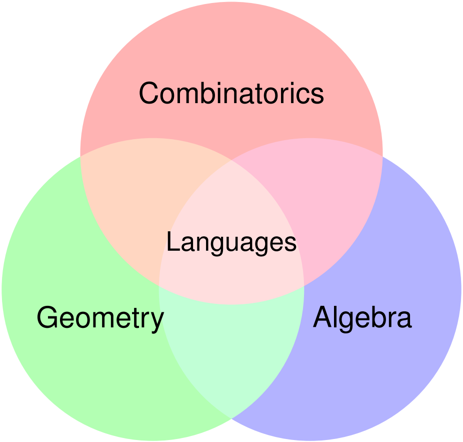

Aim of today: find out which 'conjugacy language' properties are preserved under quasi-isometries.

Formal languages

Notation:

- $X =$ finite set.
- $X^* =$ set of all finite words over X.

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Example 2

$X = \{a, b, c, \ldots, z\}$.
L = English Language (Collins Dictionary 2023).

Chomsky hierarchy

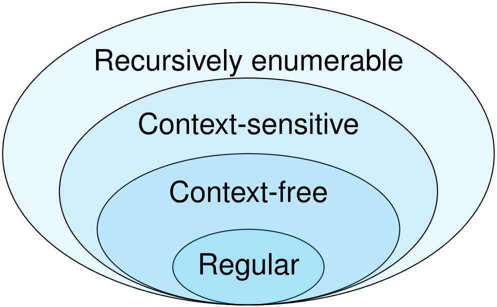

Figure: Chomsky hierarchy

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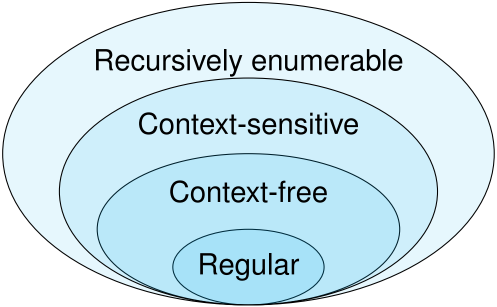

Figure: Chomsky hierarchy

Regular $\Leftrightarrow$ Finite-state automaton.
Context-free $\Leftrightarrow$ Pushdown automaton.

Finite-state automaton

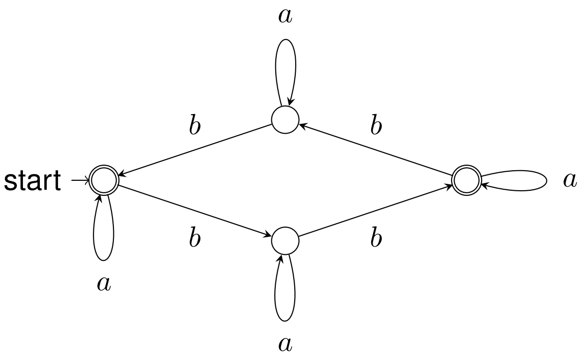

Figure: Finite-state automaton (FSA)

Finite-state automaton

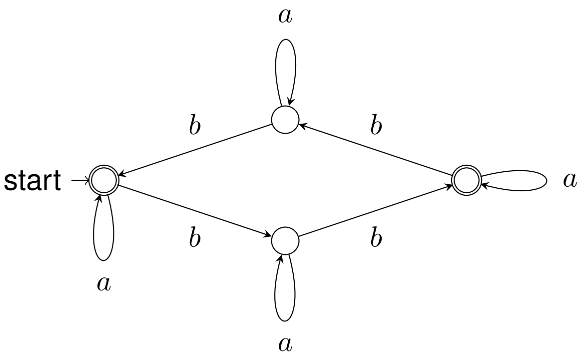

Figure: Finite-state automaton (FSA)

$X = \{a, b\}$, $L =$ all words over X which contain an even number of b's.

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- L is regular $\Rightarrow f_L(z)$ is rational.
- L is unambiguous context-free (UCF) $\Rightarrow f_L(z)$ is algebraic (Chomsky-Schützenberger).
- If $f_L(z)$ is transcendental $\Rightarrow L$ is not UCF (therefore not regular).

Languages from groups

$G = \langle X \rangle$ finitely generated group. $\pi : X^* \to G$. For $g \in G$, define:

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Conjugacy geodesic language

$$\mathsf{ConjGeo}(G, X) := \{w \in X^* \mid \ell(w) = |\pi(w)|_c\}$$

'Words which are shortest with respect to their conjugacy class'.

Conjugacy geodesics

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e.g. $w = a^{-1}bba$. Consider conjugating w by a:

$$awa^{-1} = aa^{-1}bbaa^{-1} = bb.$$

Languages from groups

For each conjugacy class c, let z_c be the shortlex least word over X representing an element of c.

Example

Let $X = \{a, b\}$. Order generators as $a < a^{-1} < b < b^{-1}$. Then for example:

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'Unique representative from each conjugacy class'.

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Ok, time for some groups!

My favourite groups!

Definition

Let Γ be a finite simple graph. The **right-angled Artin group** on Γ, denoted A_Γ, is the group defined by the following presentation:

$$A_\Gamma = \langle V(\Gamma) \mid [u, v] = 1 \text{ if and only if } \{u, v\} \in E(\Gamma) \rangle$$

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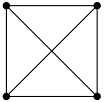
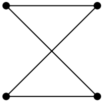
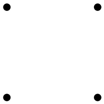


$\mathbb{Z}^4$

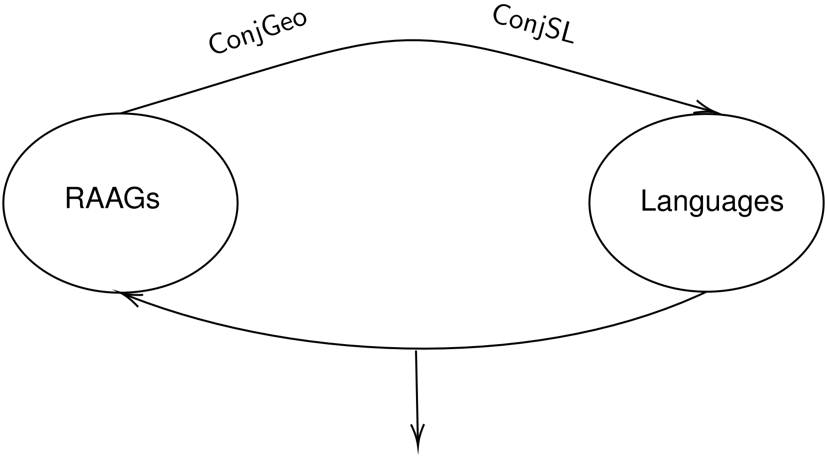


$F_2 \times F_2$



F_4

Summary



$A_\Gamma \leq G$?

Results

Group	ConjGeo	ConjSL
Word hyperbolic	Regular*[1]	Not UCF [2]
XL - type Artin	Regular [1]	Not regular [1]
Virtually abelian	PT [1]	-
Virtually cyclic	Regular*	Regular*[1]
RAAGs	Regular [3]	Not CF [C, 22]
(Some) virtual RAAGs	...	Not CF [C, 22]

$*$: true for all generating sets.
PT: Piecewise testable.
[1]: Ciobanu, Hermiller, Holt, Rees (2016).
[2]: Antolin, Ciobanu (2017).
[3]: Ciobanu, Hermiller (2013).

Results

Theorem (C, 2022)

Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \mathbb{Z}/m\mathbb{Z}$. Then there exists examples of RAAGs A_Γ such that:

1. $\mathsf{ConjGeo}(A_\phi, X)$ *is regular, or*
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Key fact: quasi-isometries don't guarantee same types of languages!

References

Yago Antolín, Laura Ciobanu (2017)
Formal Conjugacy Growth in Acylindrically Hyperbolic groups
International Mathematics Research Notices (1):121-157

Laura Ciobanu, Susan Hermiller (2013)
Conjugacy growth series and languages in groups
Transactions of the American Mathematical Society 366(5):2803-2825

Laura Ciobanu, Susan Hermiller, Derek Holt, Sarah Rees (2016)
Conjugacy languages in groups
Israel Journal of Mathematics 211(1):311-347

Gemma Crowe (2022)
Conjugacy languages in virtual graph products
https://arxiv.org/abs/2212.07111

Thank you for listening!

Any questions?