Combinatorial Moment Sequences

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partly based on joint works with Einar Steingrímsson (Strathclyde) and Slim Kammoun (Toulouse)

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Moments

Take a combinatorial sequence $(m_n)_{n\geq 0}$ with $m_0 = 1$.

Question: When is there a probability measure μ on $\mathbb{R}$ such that $\forall n$,

$$m_n = \int_{\mathbb{R}} x^n \, d\mu(x) \quad ?$$

E.g. $n! = \int_0^\infty x^n \, e^{-x} dx$

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Answer: (Hamburger, 100 years ago) μ exists iff the Hankel matrices

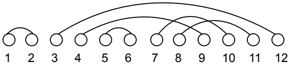
$$H_n = \begin{bmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n+1} \\ \vdots & & & \vdots \\ m_n & m_{n+1} & \dots & m_{2n} \end{bmatrix}$$

are positive semi-definite for all $n \geq 1$.

$$\int_0^\infty x^n exp(-x)\, dx \quad = \quad \#\text{ permutations on } [n]$$

$$\sum_{k \geq 0} k^n \frac{e^{-1}}{k!} \quad = \quad \#\text{ set partitions on } [n]$$

$$\int_{-\infty}^{\infty} x^{2n}\ \frac{exp(-x^2/2)}{\sqrt{2\pi}}\, dx \quad = \quad \#\text{ perfect matchings on } [2n]$$



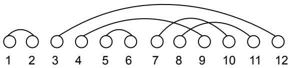
$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx \quad = \quad \#\text{ non-crossing perfect matchings on } [2n]$$

Classical CLT

Theorem

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) = 1$. Then $S_N := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \xrightarrow{d} \mathcal{N}(0,1)$. Equivalently,

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}\left(S_N^{2n-1}\right) &= 0, \\ \lim_{N\to\infty} \mathbb{E}\left(S_N^{2n}\right) &= (2n-1)!! := (2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1 \\ &= \sum_{\pi \in \mathcal{P}_2(2n)} 1 \end{aligned}$$



Proof.

Product of sums as a sum of products:

$$\mathbb{E}(S_N^k) = \frac{1}{N^{k/2}} \sum_{i(1),\ldots,i(k) \in [N]} \mathbb{E}(X_{i(1)} \cdots X_{i(k)}).$$

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- Independence $\implies$ factorization. E.g.

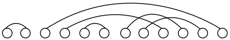
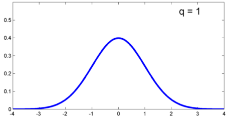
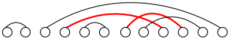
$$\mathbb{E}(X_1 X_2 X_2 X_1 X_1) = \mathbb{E}(X_1^3)\mathbb{E}(X_2^2)$$

- Independence + identical distribution $\implies$ same repetition patterns yield identical mixed moments. E.g.

$$\mathbb{E}(X_1 X_2 X_2 X_1 X_1) = \mathbb{E}(X_5 X_3 X_3 X_5 X_5)$$

- $\mathbb{E}(X) = 0 \implies$ partitions with a singleton don't contribute.
- Remaining partitions with a block of size ≥ 3 are too few ($o(N^{k/2})$). Hence, only pair partitions ($\Theta(N^{k/2})$ for k even) appear in the limit and

$$\lim_{N\to\infty} \mathbb{E}\left(S_N^{2n-1}\right) = 0, \quad \lim_{N\to\infty} \mathbb{E}\left(S_N^{2n}\right) = \sum_{\pi\in\mathcal{P}_2(2n)} 1.$$

Comm. rel.	**Law of $a_i + a_i^*$ on $\bigoplus_n H^{\otimes n}$**	**CLT & Moments**
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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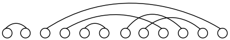
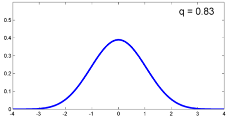
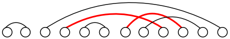
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies$ q-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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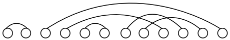
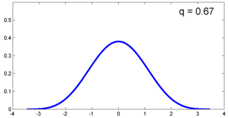
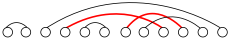
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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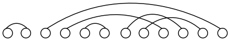
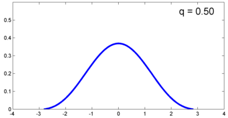
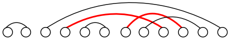
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1+\ldots+X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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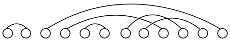
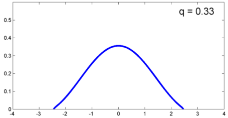
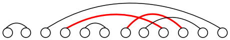
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992  $q^{\#\text{crossings}}$

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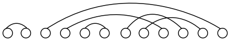
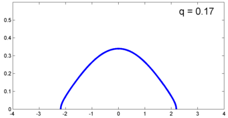
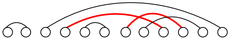
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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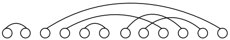
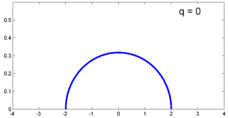
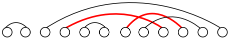
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \Longrightarrow q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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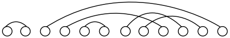
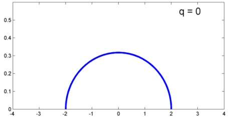
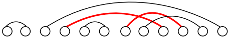
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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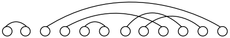
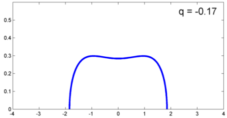
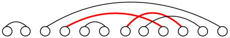
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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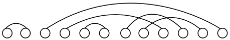
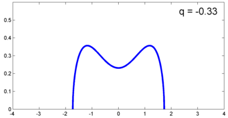
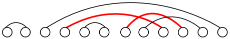
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1+\ldots+X_N}{\sqrt{N}} \Longrightarrow$ q-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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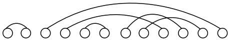
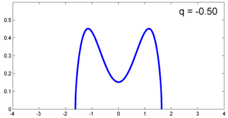
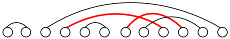
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1, 1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992  $q^{\#\text{crossings}}$

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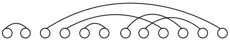
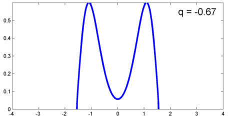
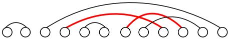
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics ‘50s/‘70s Maths ‘90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ “iid” $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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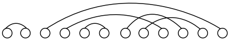
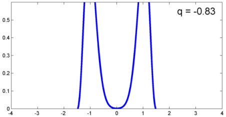
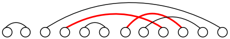
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics ‘50s/‘70s Maths ‘90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ “iid” $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

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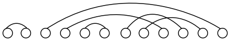
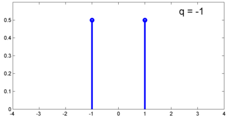
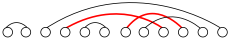
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1, 1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q$-Gauss Speicher 1992  $q^{\#\text{crossings}}$

$$a_i a_j^* - q a_j^* a_i = t^N \delta_{i,j}$$

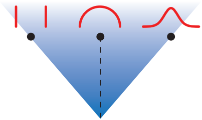
Physics '90s

Bożejko & Yoshida '06

B. 2012 JFA

'(q,t)-Gaussian'

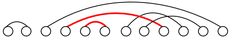
$$|q| \le t$$



$$X_i^{\epsilon} X_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j,i) X_j^{\epsilon'} X_i^{\epsilon}$$

$$\mu_{\epsilon',\epsilon}(j,i) \in \mathbb{R}$$

B. 2014 AIHP



$$q^{\#\text{crossings}} \, t^{\#\text{nestings}}$$

$$a_i a_j^* - q a_j^* a_i$$
$$= \delta_{i,j} + \alpha \langle e_i, \Pi_0 e_j \rangle q^{2N}$$

Bożejko, Ejsmont,

& Hasebe 2015

'type B' Gaussian

$$q, \alpha \in (-1,1)$$

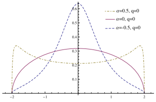


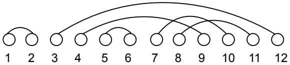
Fig. 1. $q = 0$.

$$X_i X_j = s(j,i) X_j X_i$$
$$Y_i Y_j = s(j,i) Y_j Y_i$$
$$X_i Y_j = r(j,i) Y_j X_i$$
$$s(j,i), r(j,i) \in \{-1, 1\}$$

B. - Ejsmont 2019 JMAA

Moments

$$\int_{-\infty}^{\infty} x^{2n} \frac{exp(-x^2/2)}{\sqrt{2\pi}}\, dx \quad = \quad \#\text{ perfect matchings on } [2n]$$



$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx \quad = \quad \#\text{ non-crossing perfect matchings on } [2n]$$

$$\sum_{k\geq 0} k^n \frac{e^{-1}}{k!} \quad = \quad \#\text{ set partitions on } [n]$$

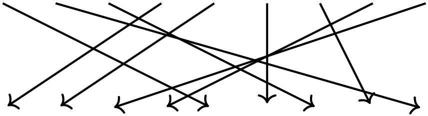
$$\int_{0}^{\infty} x^n exp(-x)\, dx \quad = \quad \#\text{ permutations on } [n]$$

(1) Which combinatorial refinements give rise to moment sequences?

(2) Which ones of those have a deeper meaning?

A combinatorial perspective on positivity (B. & Steingrímsson '21)

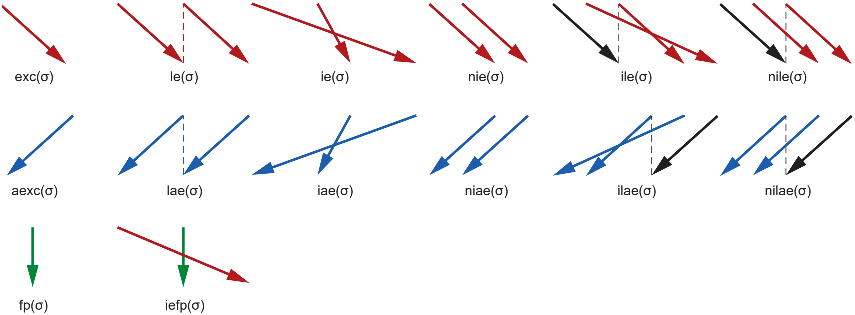
Represent $\sigma \in S_n$ in 2-line notation: i [top] $\mapsto \sigma(i)$ [bottom]



$$\sigma = 597126843$$

Definition (B. & Steingrímsson '21)

For $\sigma \in S_n$,



Theorem (B.-Steingrímsson, 2021) As formal power series,

$$\mathcal{C}(z) =$$

$$\sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)} \times \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)} z^n$$

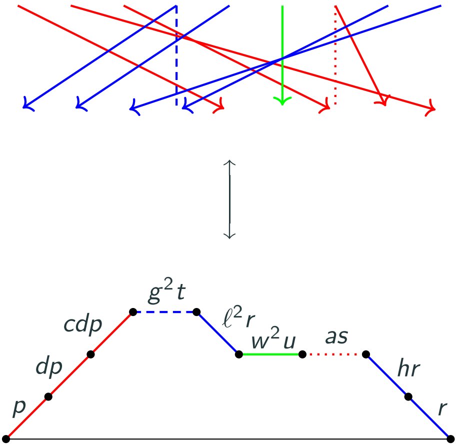
$$= \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with $\alpha_n = u\,w^n + s\,[n]_{a,b} + t\,[n]_{f,g}$, $\qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$

where $[n]_{x,y} = \sum_{k=0}^{n-1} x^k y^{n-1-k}$. (For $x \neq y$, $[n]_{x,y} = \frac{x^n - y^n}{x-y}$.)

Proof.

Exhibit a bijection:



😈 $\in$ bijection

Related to/extends:

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

Contemporaneous: Sokal & Zeng 2020 (arXiv:2003.08192, 122 p.).

😈 ∈ bijection

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Lancaster, 6-10 June 2022

Quick example

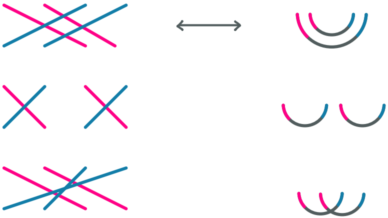
$c = s = t = u = 0$, $a = b = d = f = g = \ell = w = p = r = 1$.

Free parameter: h.

No: fixed points, linked excedances, linked antiexcedances, inversions among excedances.

Yes: $h^{\#\text{inversions among anti-excedances}}$

$n = 4$:



Another quick example

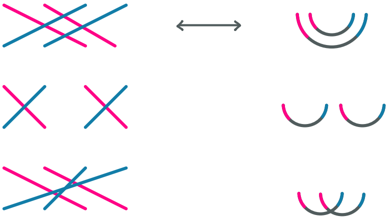
$c = s = t = u = 0$, $a = b = d = f = g = w = p = r = 1$.

Free parameters: h, ℓ.

No: fixed points, linked excedances, linked antiexcedances, inversions among excedances.

Yes: $h^{\#\text{inversions among anti-excedances}}$ $\ell^{\#\text{restricted non-inversions among anti-excedances}}$

$n = 4$:



More examples

Set partitions by # blocks (Stirling, second kind)	$\longleftrightarrow$	Poisson
Non-crossing set partitions by # blocks (Narayana)		Free Poisson (Marchenko-Pastur)
Eulerian polynomials $\sum_\sigma q^{\mathrm{des}(\sigma)}$		Geometric
Derangements		e.g. Martin & Kearney '15
Alternating permutations		Sokal '18
Little Schroeder numbers		Młotowski & Penson '13
k-arrangements		Shifted exponentials
Inv, Exc, FP on k-colored perm.		B. & Steingrímsson '21

Positivity

Fix $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$ and let

$$m_n = \sum_{\sigma \in \mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma) - \mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma) - \mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma) - \mathrm{ilae}(\sigma)}$$
$$\times \ell^{\mathrm{niae}(\sigma) - \mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma) - \mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma) - \mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}.$$

Is (m_n) a moment sequence of some positive Borel measure on $\mathbb{R}$?

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$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}} = m_0 + m_1 z + m_2 z^2 + \dots$$

with

$$\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell}$$

Answer:

If $\beta_1, \ldots, \beta_{k-1} > 0$ and $\beta_n = 0$ for all $n \geq k$, measure supported on k elements.

If $\beta_n > 0$ for all n, measure exists (need not be unique).

Answer:

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Perspective 1: Hamburger moment problem

$$H_n = \det \begin{bmatrix} m_0 & m_1 & \ldots & m_n \\ m_1 & m_2 & \ldots & m_{n+1} \\ \vdots & & & \vdots \\ m_n & m_{n+1} & \ldots & m_{2n} \end{bmatrix} = (\beta_1)^n (\beta_2)^{n-1} \ldots (\beta_{n-1})^2 \beta_n > 0$$

E.g. Partitions of $[n]$: $e_n = \sum_\pi x^{\#\text{blocks}(\pi)}$. Then, $H_n = x^{\binom{n+1}{2}} \prod_{k=1}^n k!$

E.g. Derangements of $[n]$: $d_n = \#\{\sigma \in S_n \mid \text{fp}(\sigma) = 0\}$. $H_n = \prod_{k=1}^n (k!)^2$

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}} = m_0 + m_1 z + m_2 z^2 + \dots$$

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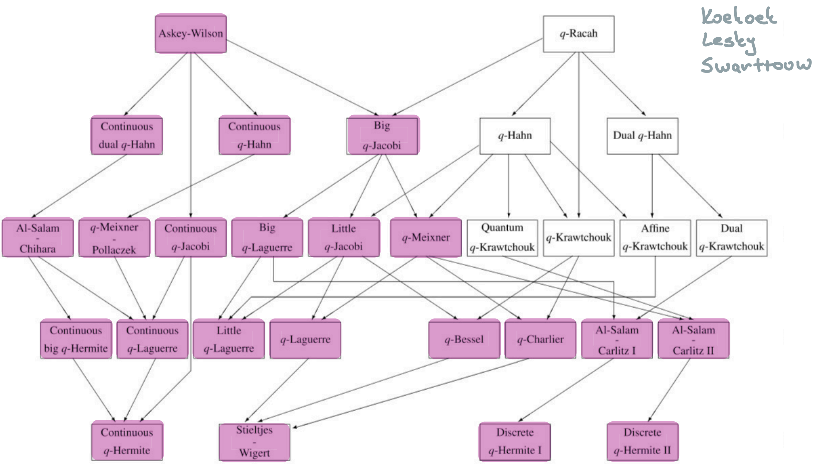
Perspective 2: Orthogonal polynomials

$$P_0(x) = 1, \quad P_1(x) = x - \alpha_0, \quad P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x)$$

Stieltjes, Favard, Shohat, ...

Consequences: (1) New Interpretations

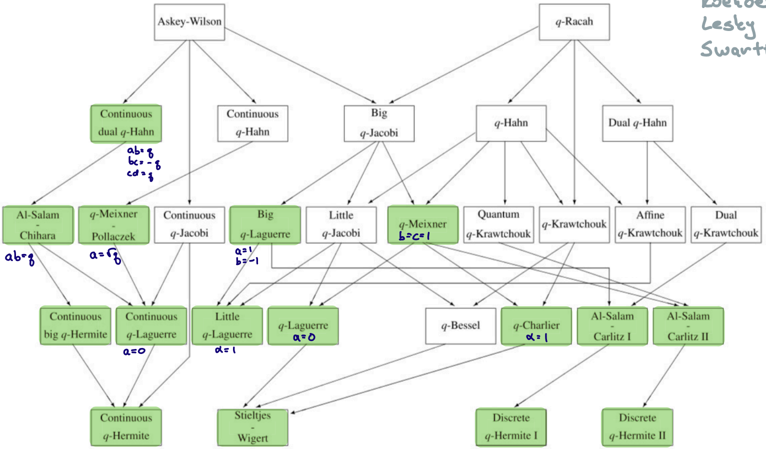


Corteel & Williams '11/'12:

$$m_n = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty} \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} \left(\frac{1-q}{2}\right)^{\ell} \frac{Z_\ell}{\Pi_{i=0}^{\ell-1}(\alpha\beta - \gamma\delta q^i)}.$$

Consequences: (1) New Interpretations

Koekoek
Lesky
Swarttouw



B. & Steingrímsson '21

$$m_n = \sum a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)}$$
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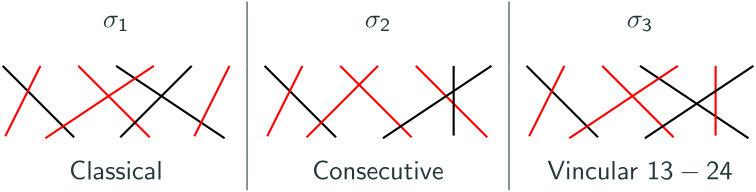
(2) New lines of attack

Definition. Permutation pattern: $\pi \in S_k$ and a containment rule

Example: $\pi = 1324$ classical, consecutive, or vincular



Let $\mathrm{Av}_\pi(n) := \#$ permutations on $[n]$ avoiding π.

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Theorem (Simion & Schmidt '85).
For any classical pattern π of length 3,

$$\mathsf{Av}_\pi(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Conjecture (Stanley-Wilf) / Theorem (Marcus & Tardos '04)
For any classical permutation pattern π,

$$(\mathsf{Av}_\pi(n))^{1/n} \to c_\pi > 0 \quad \text{as } n \to \infty.$$

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Only 3 symmetry classes for patterns of length 4:

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1234 Gessel '90 (exact) |

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Only 3 symmetry classes for patterns of length 4:

1234 Gessel '90 (exact) | 1342 Bona '97 (exact) | 1324 wide open, $c = ?$

Theorem (Rains '98). Let $\pi = 1\,2\,3\ldots k$ classical ($k \in \mathbb{N}$). Then,

$$\mathrm{Av}_\pi(n) = \mathbb{E}_{U \in \mathbb{U}(k)}(|Tr(U)^2|^n).$$

Conjecture (B. & Steingrímsson, Guttmann & Elvey Price).
For any classical pattern π, $(\mathrm{Av}_\pi(n))_n$ is a moment sequence.

Numerical evidence for patterns of length 4: Bostan, Elvey Price, Guttmann, Maillard '20

Numerical evidence for patterns of length 5: Clisby, Conway, Guttmann, Inoue '21+

$$\mathcal{C}(z) = \mathcal{C}_{a,b,c,d,f,g,h,\ell,p,r,s,t,u,w}(z) := \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with

$$\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell}.$$

Theorem (B. & Steingrímsson, '21) For any pattern of length 3 (classical, consecutive, vincular), $(\mathrm{Av}_\pi(n))_n$ is a moment sequence $\iff \sum_{n\geq 0} \mathrm{Av}_\pi(n) z^n$ is a special case of $\mathcal{C}(z)$.

Refines further:

Corollary Set $s = qx$, $p = x$, all other parameters $= 1$,

$$\mathcal{C}(z) = \sum_{n\geq 0} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{occ}_{321}(\sigma)} z^n.$$

(Previously in Elizalde '17.)

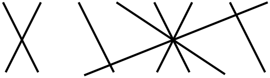
Corollary Set $b = d = g = l = q$, $s = qx$, $p = u = x$ all other parameters $= 1$,

$$\mathcal{C}(z) = \sum_{n\geq 0} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)+1} q^{\mathrm{occ}_{2-31}(\sigma)} z^n.$$

(Previously in Claesson & Mansour '02.)

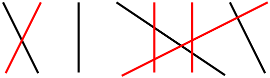
(3) Further generalizations

S_n



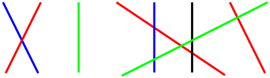
Permutations

B_n



Signed permutations

$\mathbb{Z}_k \wr S_n$



k-colored permutations

Steingrímsson ’94 defines fixed points, excedances, anti-excedances, and inversions.

Corollary (B. & Steingrímsson '21)
For $s = p = kx$, $t = r = ky$, $u = (k-1)x + q$ *(all other parameters =1),*

$$\mathcal{C}(z) = \sum_{n\geq 0} \sum_{\sigma \in S_n^k} x^{exc(\sigma)} y^{aexc(\sigma)} q^{fxpt(\sigma)} z^n.$$

(Refines further.)

Corollary (B. & Steingrímsson '21)
For $a = c = h = r = q$, $b = f = d = \ell = t = q^2$, $g = w = 0$, $p = u = 1$, $s = 2q$,

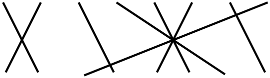
$$\mathcal{C}(z) = \sum_{n\geq 0} \sum_{\sigma \in S_n^k} q^{inv(\sigma)} z^n.$$

Recovers Biane '93 for $k = 1$.

Definition (B. & Steingrímsson '21)

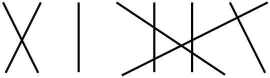
***k*-arrangement**: $\sigma \in S_n$ and a k-coloring of the fixed points in σ ($k \in \mathbb{N}_0$)

$k = 0$



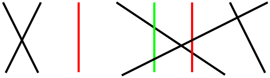
Derangements

$k = 1$



Permutations

$k = 2$



Arrangements (Comtet '74)

Decorated permutations (Postnikov '06)

Same form of CF, replace uw^n by $(u_1 + \ldots + u_k)w^n$.

$\rightsquigarrow$ Same 14 statistics + distinguishes the color of the fixed points.

$$A_k(n) := \begin{cases} 1, & n = 0 \\ \#k\text{-arrangements on } [n], & n \in \mathbb{N} \end{cases}.$$

Easy Proposition:

- $A_k(n) = nA_k(n-1) + (k-1)^n$.
- $A_k(n) = \operatorname{perm} \begin{bmatrix} k & 1 & 1 & \cdots & 1 \\ 1 & k & 1 & \cdots & 1 \\ 1 & 1 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & & 1 \\ 1 & 1 & 1 & \dots & k \end{bmatrix}.$
- $A_k(n) = \sum_{i \geq 0} \binom{n}{i} A_{k-1}(i).$

Fix a classical permutation pattern π of length 3. Let:

$$\mathrm{Av}_\pi(n;k) := \#k\text{-arrangements on } [n] \text{ avoiding } \pi.$$

Proposition (Simion & Schmidt '85)

$$\mathrm{Av}_\pi(n;1) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

Proposition (B. & Steingrímsson '21)

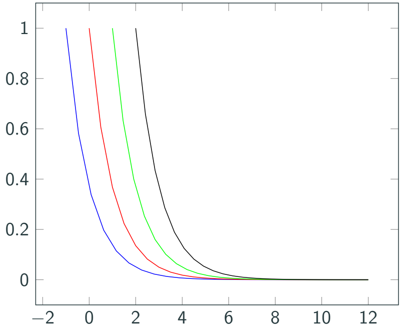
$$\mathrm{Av}_\pi(n;2) = C_{n+1}$$

~~**Conjecture** (B. & Steingrímsson '21)~~ **Theorem (Fu, Han, Lin '20)**

$$\mathrm{Av}_\pi(n;3) = C_{n+2} - 2^n$$

Proposition (B. & Steingrímsson, '21)

$$\#k\text{-arrangements on } [n] = \int_{k-1}^{\infty} x^n e^{-x+(k-1)} dx$$



Positivity previously observed for:

- $k = 0$: Martin & Kearney '15
- $k = 2$: Ardila, Rincón, Williams '16 (# positroids)

(2) Revisited: a closer look at descents and consecutive permutation patterns

Recall the **descent set** of a permutation $\sigma \in S_n$:

$$DES(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$$

The number of permutations $\#\sigma \in S_n$ with

$$DES(\sigma) = \{c_1 < c_2 < \ldots < c_k\}$$

is

$$\det\left(\binom{a_i}{b_j}\right)_{1 \le i,j \le k+1}$$

where

$(a_1, \ldots, a_{k+1}) = (c_1, \ldots, c_k, n)$ and $(b_1, \ldots, b_{k+1}) = (0, c_1, \ldots, c_k)$.
MacMahon 1908

Non-intersecting paths (Gessel & Viennot '85), determinantal point processes (Borodin, Diaconis & Fullman '10),. . .

Observation: a descent is the **consecutive pattern 21**.

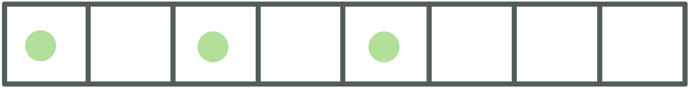
What if we keep track of occurrences of a given consecutive pattern?

Less tractable than descents!

Partial results by Elizalde & Noy '03, Elizalde & Noy '12, Beaton, Conway, Guttmann '18

Example Count $\sigma \in S_8$ with occurrences of the pattern 1324 in positions

$$1, 3, 5.$$



Example Count $\sigma \in S_8$ with occurrences of the pattern 1324 in positions

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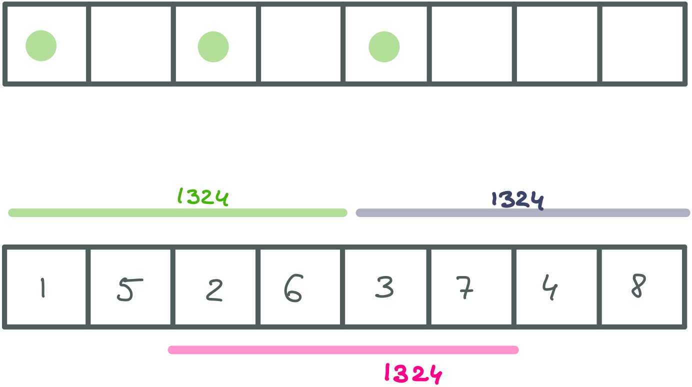

Example Count $\sigma \in S_8$ with occurrences of the pattern 1324 in positions

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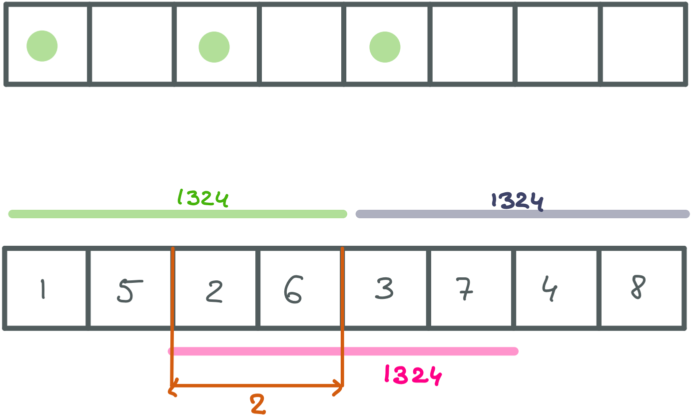

Answer (for 1324, overlaps of size 2) Catalan numbers C_{n-1}, where n is the number of occurrences.

1324 with overlaps 2: $\dfrac{1}{2n+1}\dbinom{2n}{n}$

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13425 with overlaps 2: $\frac{1}{2n+1}\binom{3n}{n}$

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1345267 with overlaps 3: $\dfrac{1}{3n+1}\dbinom{4n}{n}$

1324 with overlaps 2: $\frac{1}{2n+1}\binom{2n}{n}$

13425 with overlaps 2: $\frac{1}{2n+1}\binom{3n}{n}$

1345267 with overlaps 3: $\frac{1}{3n+1}\binom{4n}{n}$

134562789 with overlaps 4: $\frac{1}{4n+1}\binom{5n}{n}$

Result (**B., Steingrímsson, Kammoun ’21+**) Fix a consecutive permutation pattern. Fix the positions of occurrences. Obtain a general recursive formula for enumerating permutations with patterns occurring in prescribed positions.

Allows us to formulate . . .

Result (**B., Steingrímsson, Kammoun '21+**) Fix a consecutive permutation pattern. Fix the positions of occurrences. Obtain a general recursive formula for enumerating permutations with patterns occurring in prescribed positions.

Allows us to formulate . . .

Conjecture For a fixed pattern and occurrences at regular intervals, the enumerating sequence is a moment sequence.

— The "right" way to look at positivity for consecutive permutation patterns.

Conclusion (for now)

Multiparameter combinatorial frameworks can:

- unite
- distinguish
- explain
- suggest new definitions and
- new points of view on familiar things

THANK YOU

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