Permutations that separate close elements

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Joint work with Tuvi Etzion (Technion)

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Torus packings
Algebraic phrasing

For $i, j \in \mathbb{Z}_n$, let $||i, j||_n$ be the distance between $i$ and $j$ when the elements of $\mathbb{Z}_n$ are written in a circle.
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**Definition (An overlapping rectangle)**

A permutation \( \pi : \mathbb{Z}_n \to \mathbb{Z}_n \) has an \((s, k)\)-clash if there exist distinct \( i, j \in \mathbb{Z}_n \) with \( ||i, j||_n < s \) and \( ||\pi(i), \pi(j)||_n < k \).
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Definition (An overlapping rectangle)

A permutation $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ has an $(s, k)$-clash if there exist distinct $i, j \in \mathbb{Z}_n$ with $||i, j||_n < s$ and $||\pi(i), \pi(j)||_n < k$.

Definition (No overlapping rectangles)

A permutation $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is $(s, k)$-clash-free if it has no $(s, k)$-clashes.
Related work


Related work

The main question

Definition (How wide can rectangles be?)

Let \( n \) and \( k \) be fixed. Define \( \sigma(n, k) \) to be the largest \( s \) such that an \((s, k)\)-clash-free permutation \( \pi \) of \( \mathbb{Z}_n \) exists.
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Let $n$ and $k$ be fixed. Define $\sigma(n, k)$ to be the largest $s$ such that an $(s, k)$-clash-free permutation $\pi$ of $\mathbb{Z}_n$ exists.


$$\sigma(n, k) \leq \lfloor (n - 1)/k \rfloor$$
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**Proof.**

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We can’t have $sk = n$:

So $sk \leq n - 1$. \qed
Mammoliti–Simpson conjecture

Theorem (SRB, JCT-A 2023)

\[\lfloor \frac{n-1}{k} \rfloor - 1 \leq \sigma(n, k) \leq \lfloor \frac{n-1}{k} \rfloor\]

Proof. \((n, k, s) = (76, 6, 11)\).

Set \(\rho(0) = 0, \rho(1) = 12, \ldots\). \(\rho\) is \((k, s)\)-clash-free.

Set \(\pi = \rho - 1\). Then \(\pi\) is \((s, k)\)-clash-free.
# Mammoliti–Simpson conjecture

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\[
\begin{pmatrix}
0 & 12 & 24 & 36 & 48 & 60 & 72 & 8 & 20 & 32 & 44 & 56 & 68 & 4 & 16 & 28 & 40 & 52 & 64 \\
1 & 13 & 25 & 37 & 49 & 61 & 73 & 9 & 21 & 33 & 45 & 57 & 69 & 5 & 17 & 29 & 41 & 53 & 65 \\
2 & 14 & 26 & 38 & 50 & 62 & 74 & 10 & 22 & 34 & 46 & 58 & 70 & 6 & 18 & 30 & 42 & 54 & 66 \\
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Then \(\pi\) is \((s, k)\)-clash-free.
Theorem (SRB–Etzion, 2023+)

Let $n$ and $k$ be fixed positive integers, with $k < n$. Write $s = \lfloor \frac{n-1}{k} \rfloor$, so $n = sk + r$ where $1 \leq r \leq k$.

Define $d_k = \gcd(n, k)$ and $d_s = \gcd(n, s)$.

If $r \geq s$ or $k = r$, then $\sigma(n, k) = \lfloor \frac{n-1}{k} \rfloor$.

If $r < s$ and $r < k$ and $d_s d_k$ divides $n$, then $\sigma(n, k) = \lfloor \frac{n-1}{k} \rfloor$.

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Threads cannot change direction:
A sketch proof 2

Threads must be periodic, giving the condition that $d_sd_k$ divides $n$. 
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Can classify permutations by *jumpers*: two sequences determining sizes of gaps.
An \((s, k, n)\)-jumper is a pair \(((a_i), (b_i))\) of sequences of integers with the following properties:

1. \((a_i)\) has period dividing \(d_s\), and \((b_i)\) has period dividing \(d_k\).
2. We have \(1 \leq a_i < s\) and \(1 \leq b_i < k\) for \(i \geq 0\).
3. The \(d_k\) partial sums \(\sum_{i=0}^{\ell-1} b_i\) where \(0 \leq \ell < d_s\) are distinct modulo \(d_k\). Moreover, \(d_s d_k\) divides \(\sigma_b\) where \(\sigma_b = \sum_{i=0}^{d_k-1} b_i\).
4. The \(d_s\) partial sums \(\sum_{i=0}^{m-1} a_i\) where \(0 \leq m < d_s\) are distinct modulo \(d_s\). Moreover, \(d_s d_k\) divides \(\sigma_a\) where \(\sigma_a = \sum_{i=0}^{d_s-1} a_i\).
5. Defining \(\sigma_a\) and \(\sigma_b\) as above, \(\sigma_a \sigma_b = d_s d_k r\).
The classification

**Theorem**

Let \( n \) and \( k \) be fixed integers with \( k < n \). Set \( s = \lfloor (n - 1)/k \rfloor \), and define \( r \) by \( n = sk + r \) for \( 1 \leq r \leq k \). Define \( d_s = \gcd(n, s) \) and \( d_k = \gcd(n, k) \).

Assume that \( r < k \) and \( r < s \). Furthermore, suppose that \( d_s d_k \) divides \( n \).
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Let $n$ and $k$ be fixed integers with $k < n$. Set $s = \lceil (n - 1)/k \rceil$, and define $r$ by $n = sk + r$ for $1 \leq r \leq k$. Define $d_s = \gcd(n, s)$ and $d_k = \gcd(n, k)$. Assume that $r < k$ and $r < s$. Furthermore, suppose that $d_s d_k$ divides $n$.

There is a bijection between the set of clockwise $(s, k)$-clash-free permutations with $\pi(0) = 0$ and the set $J(s, k, n)$ of $(s, k, n)$-jumpers.
Thanks!