Permutations that separate close elements

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Joint work with Tuvi Etzion (Technion)

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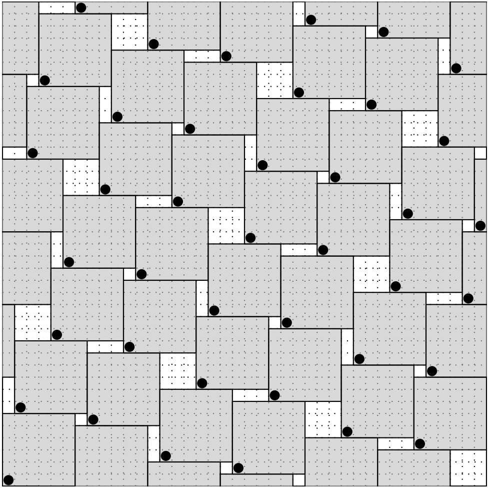
Torus packings



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For $i, j \in \mathbb{Z}_n$, let $||i, j||_n$ be the distance between i and j when the elements of $\mathbb{Z}_n$ are written in a circle.

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Definition (No overlapping rectangles)

A permutation $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ is (s, k)-*clash-free* if it has no (s, k)-clashes.

Related work

- Generalisations of $k = 2$ case: cyclic matching sequencability for graphs: Alspach, *Bull. ICA* 2008, Brualdi–Kiernan–Meyer, *Australas. J. Comb.* 2012; Kreher–Pastine–Tollefson, *Australas. J. Comb.* 2015.

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- Packing diamonds rather than rectangles (large distance in the Manhattan metric): Aspvell–Liang *Stanford Tech. Report* 1980; Bevan–Homberger–Tenner *JCT-A* 2018; SRB–Homberger–Winkler *JCT-A* 2019.

The main question

Definition (How wide can rectangles be?)

Let n and k be fixed. Define $\sigma(n,k)$ to be the largest s such that an (s,k)-clash-free permutation π of $\mathbb{Z}_n$ exists.

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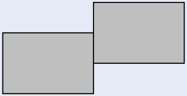

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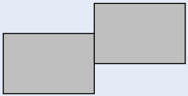

So $sk \leq n - 1$. □

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Theorem (SRB, JCT-A 2023)

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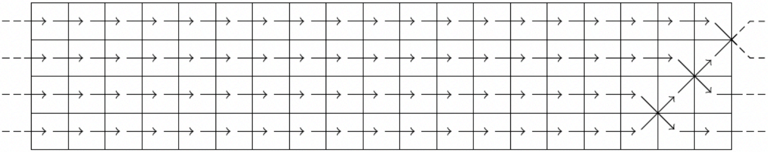

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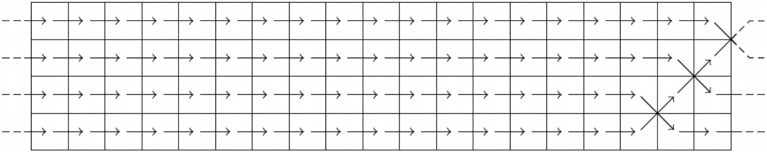

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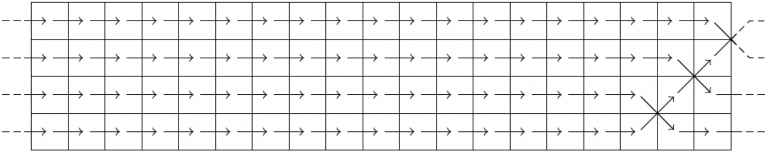

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Theorem (SRB–Etzion, 2023+)

Let n and k be fixed positive integers, with $k < n$. Write $s = \lfloor (n-1)/k \rfloor$, so $n = sk + r$ where $1 \leq r \leq k$.

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- *If $r < s$ and $r < k$ and $d_s d_k$ does not divide n, then $\sigma(n,k) = \lfloor (n-1)/k \rfloor - 1$.*

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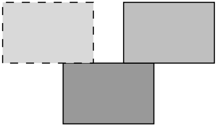

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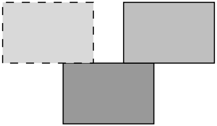

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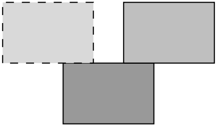

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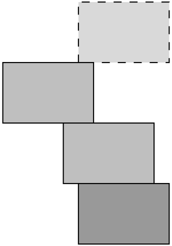
Threads cannot change direction:



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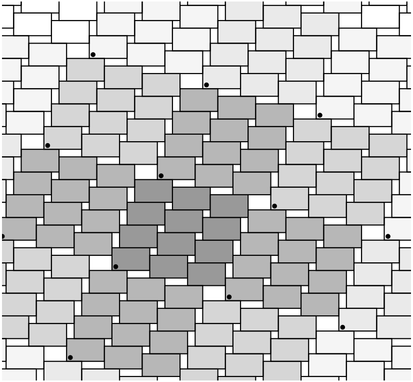

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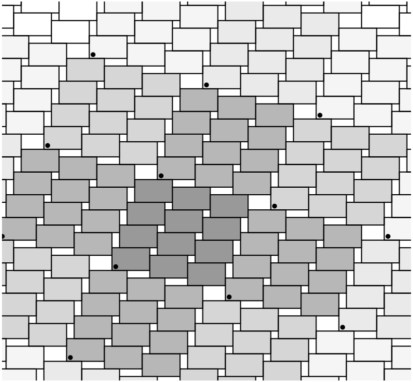

Can classify permutations by jumpers: two sequences determining sizes of gaps.

Jumpers

Definition

An (s, k, n)-jumper is a pair $\big((a_i), (b_i)\big)$ of sequences of integers with the following properties:

1. (a_i) has period dividing d_s, and (b_i) has period dividing d_k.
2. We have $1 \leq a_i < s$ and $1 \leq b_i < k$ for $i \geq 0$.
3. The d_k partial sums $\sum_{i=0}^{\ell-1} b_i$ where $0 \leq \ell < d_s$ are distinct modulo d_k. Moreover, $d_s d_k$ divides σ_b where $\sigma_b = \sum_{i=0}^{d_k-1} b_i$.
4. The d_s partial sums $\sum_{i=0}^{m-1} a_i$ where $0 \leq m < d_s$ are distinct modulo d_s. Moreover, $d_s d_k$ divides σ_a where $\sigma_a = \sum_{i=0}^{d_s-1} a_i$.
5. Defining σ_a and σ_b as above, $\sigma_a \sigma_b = d_s d_k r$.

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Theorem

Let n and k be fixed integers with $k < n$. Set $s = \lfloor (n-1)/k \rfloor$, and define r by $n = sk + r$ for $1 \leq r \leq k$. Define $d_s = \gcd(n, s)$ and $d_k = \gcd(n, k)$. Assume that $r < k$ and $r < s$. Futhermore, suppose that $d_s d_k$ divides n.

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Thanks!