New Refinements of a Classical Formula in Consecutive Pattern Avoidance

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Let $\mathrm{Av}_n(\pi)$ denote the set of all permutations of length n avoiding π as a *consecutive* pattern. In 1962, David and Barton [1] gave the exponential generating function formula

$$\sum_{n=0}^{\infty} |\mathrm{Av}_n(12\cdots m)| \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \Big(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!}\Big)\right]^{-1}$$

for counting permutations avoiding the monotone consecutive pattern $12\cdots m$. David and Barton's formula can be proven in a multitude of ways, one of which is the celebrated cluster method of Goulden and Jackson [5], adapted for permutations by Elizalde and Noy [3]. More recently, Elizalde [2] gave a q-analogue of the Goulden–Jackson cluster method which he then used to derive a refinement of David and Barton's formula that also keeps track of the inversion number.

In this talk, we present several refinements of David and Barton's formula for "inverse statistics". Given a permutation statistic st, let us define its *inverse statistic* ist by $\mathrm{ist}(\pi) = \mathrm{st}(\pi^{-1})$. For example, if des is the descent number statistic, then the inverse descent number is given by $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1})$. One of our new results is the following. Define $A_{m,n}(t,q)$ by

$$A_{m,n}(t,q) \coloneqq \sum_{\pi \in \mathrm{Av}_n(12\cdots m)} t^{\mathrm{ides}(\pi)+1} q^{\mathrm{imaj}(\pi)}$$

for $n \geq 1$ and $A_{m,0}(t,q) \coloneqq 1$. Then if $m \geq 2$, we have

$$\sum_{n=0}^{\infty} \frac{A_{m,n}(t,q)}{\prod_{i=0}^{n}(1-tq^i)} x^n = 1 + \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1}_q x^{jm} - \binom{k+jm}{k-1}_q x^{jm+1} \right)\right]^{-1} t^k.$$

We also prove analogous formulas for the "inverse peak number" and "inverse left peak number" statistics, and present real-rootedness conjectures for the associated polynomials.

To derive these new formulas, we first prove a lifting of the Goulden–Jackson cluster method to the Malvenuto–Reutenauer algebra **FQSym** (also known as the algebra of free quasisymmetric functions). By applying appropriate homomorphisms to the cluster method in **FQSym**, we can recover both the cluster method for permutations as well as Elizalde's q-analogue, so the cluster method in **FQSym** can be thought of as a unifying generalization of both. Additional homomorphisms can be obtained from the theory of shuffle-compatibility developed by Gessel and Zhuang [4], as every "shuffle-compatible" permutation statistic st induces a homomorphism on **FQSym** which can be used to count permutations by ist. By applying these new homomorphisms to the result of applying the cluster method in **FQSym** to the monotone consecutive pattern $12\cdots m$, we obtain our formulas.

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[5] I. P. Goulden and D. M. Jackson. An inversion theorem for cluster decompositions of sequences with distinguished subsequences. *J. London Math. Soc. (2)*, 20(3):567–576, 1979.