A simple proof of a CLT for vincular permutation patterns for conjugation invariant permutations[1]

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The number of occurrences of any fixed vincular permutation pattern in a uniform permutation is known to satisfy a central limit theorem. Using a comparaison technique, we extend this result to other non-uniform permutations. The technique can be used for other statistics.

1 Main result

A random permutation σ_n is called conjugation invariant if its law is conjugation invariant i.e. if $\rho \circ \sigma_n \circ \rho^{-1} \overset{d}{=} \sigma_n$ for any $\rho \in \mathfrak{S}_n$. For example, the uniform permutation, a uniform cyclic permutation and (generalized) Ewens permutations are classic examples of conjugation invariant permutations.

A vincular pattern of size p is a couple (τ, X) such that $\tau \in \mathfrak{S}_p$ and $X \subset [p-1]$. Given $\sigma \in \mathfrak{S}_n$, an occurrence of (τ, X) is a list $i_1 < \cdots < i_p$ such that

- $i_{x+1} = i_x + 1$ for any $x \in X$.
- $(\sigma(i_1), \ldots, \sigma(i_p))$ is in the same relative order as $(\tau(i_1), \ldots, \tau(i_p))$.

We denote by $\mathcal{N}_{(\tau,X)}(\sigma)$ the number of occurrences of (τ, X) in σ.

For the uniform case, Bóna [2], Janson et al. [5] and Hofer [4] proved respectively a CLT for monotone, classic and vincular patterns. Féray [3] gave a generalization for the Ewens distribution. In particular, Hofer [4] proved that for any $\tau \in \mathfrak{S}_p$ and any $X \subset [p-1]$,

$$\frac{\mathcal{N}_{(\tau,X)}(\sigma_{unif,n}) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, V_{\tau,X}). \tag{1}$$

Here, $\sigma_{unif,n}$ is a uniform permutation of size n, $q = \mathrm{card}(X)$ and $V_{\tau,X} > 0$. We prove the folowing.

[1] The full paper is available in https://arxiv.org/pdf/2012.05845.pdf

	$\sigma_{unif,3}$	$T(\sigma_{unif,3})$	$T^2(\sigma_{unif,3})$
Id	1/6	0	0
$(1,2)$	1/6	1/18	0
$(1,3)$	1/6	1/18	0
$(2,3)$	1/6	1/18	0
$(1,2,3)$	1/6	5/12	1/2
$(1,3,2)$	1/6	5/12	1/2

Table 1: Transitions for $\sigma_{unif,3}$

Proposition 1. *Suppose that for any $n \geq 1$, σ_n is conjugation invariant random permutation of $\mathfrak{S}_n$ and the sequence of number of cycles $(\#(\sigma_n))_{n\geq 1}$ satisfies*

$$\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0 \tag{2}$$

Then, for any $\tau \in \mathfrak{S}_p$ and any $X \subset [p-1]$

$$\frac{\mathcal{N}_{(\tau,X)}(\sigma_n) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, V_{\tau,X}). \tag{3}$$

We give the proof of this result in the next section. We give then an idea of a generalization to other statistics on permutations using the same kind of proofs.

2 Idea of proof

The proof uses a coupling argument. We will define a Markov chain with an Ewens stationary measure and such that conjugation invariant random permutations with few cycles are converges to the stationary measure rapidly. Formally, let ρ_n be a conjugation invariant random permutation. The idea is to modify ρ_n to obtain a uniform cyclic permutation. We define the following Markov operator T:

- If the realization σ of ρ_n has one cycle, σ remains unchanged ($T(\sigma) = \sigma$).
- Otherwise, we choose a couple (i, j) uniformly from the nonempty set

  $$\{(i,j) : j \notin \mathcal{C}_i(\sigma)\}$$

  and we take $T(\sigma) = \sigma \circ (i, j)$. Here $\mathcal{C}_i(\sigma)$ is the cycle of σ containing i.

For example, for $n = 3$, transition probabilities of T are given in Figure 1. We denote by $T^k(\rho_n)$ the random permutation obtained after applying k times the operator T. It is the random permutation obtained after k steps of the uniform random walk on $\mathcal{G}_{\mathfrak{S}_n}$ with initial state ρ_n. Table 1 sums up the evolution of the random walk if we start from the uniform distribution on $\mathfrak{S}_3$. We have then the following :

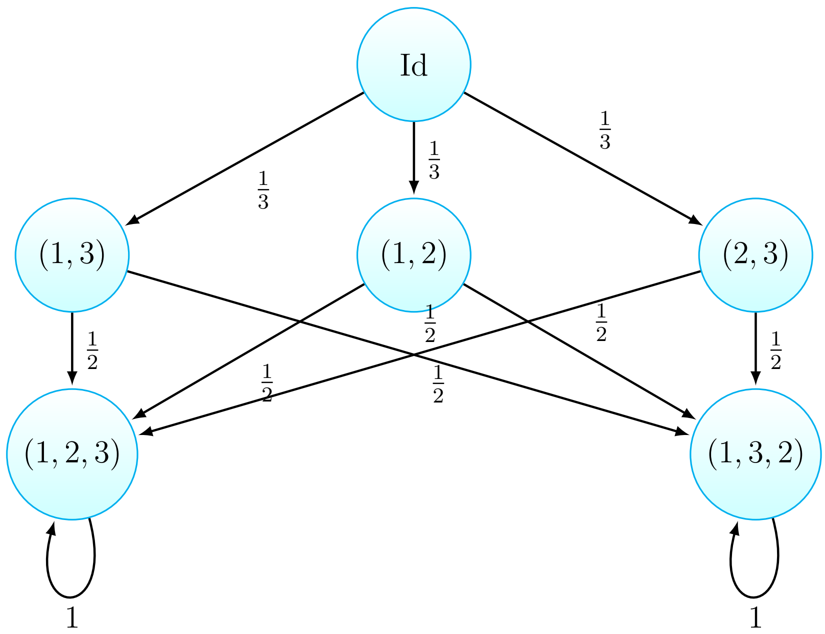

Figure 1: The transition probabilities of T for $n=3$

- If σ_n is conjugation invariant then $T(\sigma_n)$ is conjugation invariant and $\#(T(\sigma_n)) = \max(\#(T(\sigma_n))-1,1)$. Consequently, $T^{n-1}(\rho_n) \stackrel{d}{=} \sigma_{Ew,0,n}$. Where $\sigma_{Ew,0,n}$ is a uniform cyclic permutation of length n.
- Almost surely,

$$|\mathcal{N}_{(\tau,X)}(\rho_n) - \mathcal{N}_{(\tau,X)}(T^{n-1}(\rho_n))| \leq n^{p-q}\#(\rho_n).$$

Choosing first ρ_n a uniform permutation, we obtain that (1) is equivalent to (3) for $\sigma_n = \sigma_{Ew,0,n}$. In a second step, we choose ρ_n a conjugation invariant random permutation satisfying (2). In the case, the convergence in (3) (for $\sigma_n = \rho_n$) is again equivalent to the same convergence in the particle case $\sigma_n = \sigma_{Ew,0,n}$ which concludes the proof.

3 Generalization

This technique is not specific to permutation patterns. Given $n \geq 1$ and $E \subset \mathfrak{S}_n$, we define

$$\text{next}(E) := \{\rho \circ (i,j); \rho \in E,\ \#(\rho \circ (i,j)) = \#(\rho) - 1\} \cup \{\rho \in E; \#(\rho) = 1\}$$

and

$$\text{final}(\sigma) := \begin{cases} \text{next}^{\#(\sigma)-1}(\{\sigma\}) & \text{if } \#(\sigma) > 1 \\ \{\sigma\} & \text{otherwise} \end{cases}.$$

In other words, next(E) is the set of permutations obtained by concatenating, if possible, two cycles of some $\sigma \in E$, and final(σ) is the set of permutations obtained by concatenating all the cycles of σ. In particular,

$$\text{final}(\sigma) \subset \mathfrak{S}_n^0 := \{\sigma \in \mathfrak{S}_n;\ \#(\sigma) = 1\}.$$

Let f be a function defined on $\mathfrak{S}_\infty := \cup_{i=1}^{\infty}\mathfrak{S}_n$ and taking its values in some metric space (F, d_F), for example $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{R}^d$. We define for $1 \leq k \leq n$,

$$\varepsilon'_{n,k}(f) := \max_{\sigma\in\mathfrak{S}_n, \#(\sigma)=k} \max_{\rho\in\text{final}(\sigma)} d_F(f(\sigma), f(\rho)).$$

We present now our main result.

Theorem 2. *Assume that for any $n \geq 1$, (σ_n) and $(\sigma_{ref,n})$ are conjugation invariant permutations of size n. Suppose that there exists $x \in F$ such that*

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x, \tag{4}$$

$$\varepsilon'_{n,\#(\sigma_{ref,n})}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{5}$$

$$\text{and that} \quad \varepsilon'_{n,\#(\sigma_n)}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{6}$$

Then

$$f(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} x. \tag{7}$$

Moreover, if the assumptions (4)–(6) hold true for the $\mathbb{L}^p$ convergence for some $p \geq 1$ instead of the convergence in probability, then so does (7).

When $F = \mathbb{R}^d$, we obtain also the convergence in distribution.

Theorem 3. *Assume that $F = \mathbb{R}^d$ and that for any $n \geq 1$, (σ_n) and $(\sigma_{ref,n})$ are conjugation invariant permutations of size n. Suppose that (5) and (6) hold true and that there exists a random variable X supported on F such that*

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{d} X.$$

Then

$$f(\sigma_n) \xrightarrow[n\to\infty]{d} X.$$

This result can be applied to many statistics including the descent process, the shape of a permutation by RSK, the number of exceedences and the longest increasing (decreasing, alternating, common) subsequence. We detailed those applications in the full version of this work. We give here only one example to illustrate this result.

Given $\sigma \in \mathfrak{S}_n$, a subsequence $(\sigma(i_1), \ldots, \sigma(i_k))$ is an increasing (resp. decreasing) subsequence of σ of length k if $i_1 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$ (resp. $\sigma(i_1) > \cdots > \sigma(i_k)$). We denote by $\text{LIS}(\sigma)$ (resp. $\text{LDS}(\sigma)$) the length of the longest increasing (resp. decreasing) subsequence of σ. For example,

$$\text{if} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}, \text{LIS}(\sigma) = 2 \text{ and } \text{LDS}(\sigma) = 4.$$

Corollary 4. *Suppose that for any $n \geq 1$, σ_n is conjugation invariant random permutation of $\mathfrak{S}_n$ and*

$$\frac{\#(\sigma_n)}{\sqrt[6]{n}} \xrightarrow[n\to\infty]{d} 0.$$

Then,

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

where F_2 is the cumulative distribution function of the GUE Tracy-Widom distribution.

This result generalizes that of Baik, Deift and Johansson[1] who proved these fluctuations for the uniform case.

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