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Closed paths whose steps are roots of unity

Gilbert Labelle[1] and Annie Lacasse[2]

[1]*LaCIM et Dép. de Mathématiques, UQAM, Montréal, Québec, Canada*
[2]*LIRMM, Montpellier, France*

Abstract. We give explicit formulas for the number $U_n(N)$ of closed polygonal paths of length N (starting from the origin) whose steps are n^{th} roots of unity, as well as asymptotic expressions for these numbers when $N \to \infty$. We also prove that the sequences $(U_n(N))_{N\geq 0}$ are P-recursive for each fixed $n \geq 1$ and leave open the problem of determining the values of N for which the *dual* sequences $(U_n(N))_{n\geq 1}$ are P-recursive.

Résumé. Nous donnons des formules explicites pour le nombre $U_n(N)$ de chemins polygonaux fermés de longueur N (débutant à l'origine) dont les pas sont des racines n-ièmes de l'unité, ainsi que des expressions asymptotiques pour ces nombres lorsque $N \to \infty$. Nous démontrons aussi que les suites $(U_n(N))_{N\geq 0}$ sont P-récursives pour chaque $n \geq 1$ fixé et laissons ouvert le problème de déterminer les valeurs de N pour lesquelles les suites *duales* $(U_n(N))_{n\geq 1}$ sont P-récursives.

Keywords: closed polygonal paths, roots of unity, P-recursive, asymptotics

1 Introduction

The subject of random walks is classical and appears in many areas of mathematics, physics and computer science (see, for example, $http://en.wikipedia.org/wiki/Random\_walks$). In this paper we combinatorially analyse a new type of closed random walks in the complex plane — a kind of restricted Brownian motion — whose steps are given by n^{th}-roots of unity. For $n \geq 1$, let $\Omega_n = \{1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}\}$ be the set of all n-th roots of unity, where $\omega_n = \exp(2\pi i/n) \in \mathbb{C}$. A polygonal path of length N, starting at the origin in the complex plane, whose steps are n-th roots of unity can be encoded by the sequence $w = [\omega_n^{k_1}, \dots, \omega_n^{k_N}]$ of its successive steps, $\omega_n^{k_j} \in \Omega_n, j = 1, \dots, N$. For $\nu = 0, \dots, n-1$, let m_ν be the number of times that ω_n^ν appears in w. We call the sequence $\vec{m} = [m_0, \dots, m_{n-1}]$ the *type* of w, and write $\vec{m} = \text{type}(w)$. Of course, the path w is closed if and only if $\omega_n^{k_1} + \cdots + \omega_n^{k_N} = 0$ if and only if

$$m_0 + m_1\omega_n + m_2\omega_n^2 + \cdots + m_{n-1}\omega_n^{n-1} = 0. \tag{1.1}$$

We call a sequence $\vec{m} = [m_0, m_1, \dots, m_{n-1}] \in \mathbb{N}^n$ *admissible* if (1.1) is satisfied. Figure 1 shows a closed pentagon made of 18-th roots of unity encoded by $[\omega_{18}^3, \omega_{18}^{11}, \omega_{18}^5, \omega_{18}^{12}, \omega_{18}^{17}]$ and a closed 11-gon made of 14-th roots of unity encoded by $[\omega_{14}^{12}, \omega_{14}, \omega_{14}^4, \omega_{14}^5, \omega_{14}^7, \omega_{14}^5\omega_{14}^{11}, \omega_{14}^{11}, \omega_{14}^9, \omega_{14}^3, \omega_{14}^{13}]$.

Clearly, the number of closed paths, of length N, with admissible type $\vec{m}$ is given by the multinomial coefficient $N!/m_0!m_1! \dots m_{n-1}!$. This implies that the number $U_n(N)$ of closed polygonal paths of

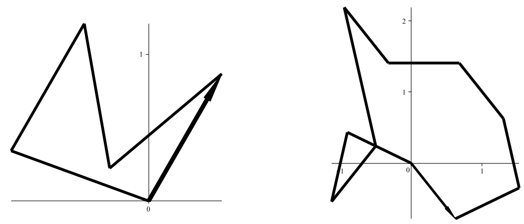

Fig. 1: Pentagon and 11-gon made of 18-th and 14-th roots of unity.

length N whose steps are n-th roots of unity is given by the formula

$$U_n(N) = \sum_{\substack{\vec{m}\,:\,\mathrm{admissible} \\ m_0+\cdots+m_{n-1}=N}} \frac{N!}{m_0!m_1!\cdots m_{n-1}!}. \tag{1.2}$$

In Section 2, we characterize admissibility and express the numbers $U_n(N)$ as *constant term* extractions in suitable rational expressions. We also give a formula from which the computation of the numbers $U_n(N)$ can be reduced to the computation of the numbers $U_q(N')$, where $N' \leq N$ and q is a suitable divisor of n. Section 3 is devoted to an analysis of recursive and asymptotic properties of the numbers $U_n(N)$. Finally, some tables are given.

2 Constant term and reduction formulas

To take advantage of formula (1.2) for $U_n(N)$ on a symbolic algebra system, we state first a simple characterization of admissibility for a sequence $\vec{m} \in \mathbb{N}^n$. This is done using the classical cyclotomic polynomials $\Phi_n(z) = \prod(z-\omega)$, where ω runs through the primitive n-th roots of unity. Equivalently, this means that $\omega = \exp(2k\pi i/n)$, where $1 \leq k \leq n$ and $\mathrm{GCD}(n,k) = 1$. Since $z^n - 1 = \prod_{d|n} \Phi_d(z)$, Moebius inversion implies that $\Phi_n(z) = \prod_{d|n}(x^d - 1)^{\mu(n/d)}$, where μ denotes the Moebius function. This shows that $\Phi_n(z)$ is a monic polynomial in $\mathbb{Z}[z]$ of degree $\varphi(n)$, the Euler function of n. The following very easy, but basic lemma characterizes admissibility.

Lemma 2.1 (criteria for admissibility). *For $n \geq 1$, the sequence $\vec{m} = [m_0, \ldots, m_{n-1}] \in \mathbb{N}^n$ is admissible if and only if the cyclotomic polynomial $\Phi_n(z)$ divides the polynomial*

$$P_{\vec{m}}(z) = m_0 + m_1 z + \cdots + m_{n-1} z^{n-1}.$$

Proof: Consider the euclidean division of $P_{\vec{m}}(z)$ by $\Phi_n(z)$ in the ring $\mathbb{Z}[z]$:

$$P_{\vec{m}}(z) = \Phi_n(z) Q_{\vec{m}}(z) + R_{\vec{m}}(z), \tag{2.1}$$

where $\deg R_{\vec{m}}(z) < \deg \Phi_n(z) = \varphi(n)$. Since $\Phi_n(\omega_n) = 0$ this shows that $\vec{m}$ is admissible if and only if $P_{\vec{m}}(\omega_n) = 0$ if and only if $R_{\vec{m}}(\omega_n) = 0$. But $R_{\vec{m}}(\omega_n) = 0$ if and only if $R_{\vec{m}}(z) = 0$ identically since $\Phi_n(z)$ is known to be the minimal polynomial of any of its roots and $\deg R_{\vec{m}} < \deg \Phi_n$. □

Euclidean division shows that the coefficients of $R_{\vec{m}}(z)$ are $\mathbb{Z}$-linear combinations $l_k(m_0,\ldots,m_{n-1})$ of the m_i's. Hence, $\vec{m}$ is admissible if and only if $l_k(m_0,\ldots,m_{n-1})=0$ for $k=0,\ldots,\varphi(n)-1$. Table 1, made using the *rem* command in Maple gives the values of the l_k's for $n=1,\ldots,20$. For example, for $n=6$, $\varphi(n)=2$ and using Table 1, formula (1.2) takes the form

$$U_6(N)=\sum_{\substack{m_0+\cdots+m_5=N\\ m_0+m_5=m_2+m_3\\ m_4+m_5=m_1+m_2}}\frac{N!}{m_0!\cdots m_5!}.$$

Note that, by the multinomial formula, this is equivalent to the following *constant term* formula

$$U_6(N)=\mathrm{CT}((t_1+t_2+\frac{t_1}{t_2}+\frac{t_2}{t_1}+t_1^{-1}+t_2^{-1})^N),$$

where $\mathrm{CT}(L(t_1,t_2,\ldots))$ denotes the constant term of the full expansion of L as a Laurent series in $t_1,t_2,\ldots$. This is generalized as follows.

Theorem 2.2 *There is a Laurent polynomial, $\Lambda_n(t_1,\ldots,t_{\varphi(n)})$, such that $U_n(N)=\mathrm{CT}(\Lambda_n(t_1,\ldots,t_{\varphi(n)})^N)$. Moreover, $\Lambda_n(t_1,\ldots,t_{\varphi(n)})$ is computed as follows. Let $m_0+\cdots+m_{n-1}z^{n-1}=\Phi_n(z)Q(z)+R(z)$, where the remainder is $R(z)=\sum_{k=0}^{\varphi(n)-1}l_k(m_0,\ldots,m_{n-1})z^k$, with $l_k(m_0,\ldots,m_{n-1})=\sum_{i=0}^{n-1}c_{k,i}m_i$, $c_{k,i}\in\mathbb{Z}$, $k=0,\ldots,\varphi(n)-1$. Then,*

$$\Lambda_n(t_1,\ldots,t_{\varphi(n)})=\sum_{j=0}^{n-1}t_1^{c_{0,j}}t_2^{c_{1,j}}t_3^{c_{2,j}}\ldots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}. \tag{2.2}$$

Proof: By the multinomial theorem,

$$\begin{aligned}&\left(\sum_{j=0}^{n-1}t_1^{c_{0,j}}\ldots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}\right)^N\\ &=\sum_{m_0+\cdots+m_{n-1}=N}\frac{N!}{m_0!\cdots m_{n-1}!}\left(t_1^{c_{0,0}}\ldots t_{\varphi(n)}^{c_{\varphi(n)-1,0}}\right)^{m_0}\cdots\left(t_1^{c_{0,n-1}}\ldots t_{\varphi(n)}^{c_{\varphi(n)-1,n-1}}\right)^{m_{n-1}}\\ &=\sum_{m_0+\cdots+m_{n-1}=N}\frac{N!}{m_0!\cdots m_{n-1}!}t_1^{l_0(m_0,\ldots,m_{n-1})}\ldots t_{\varphi(n)}^{l_{\varphi(n)-1}(m_0,\ldots,m_{n-1})}.\end{aligned}$$

The result follows since the constant term is given by taking the sum of the terms corresponding to the exponents $l_k=0$ for $k=0,\ldots,\varphi(n)-1$. □

Table 2 gives the rational functions $\Lambda_n(t_1,\ldots,t_{\varphi(n)})$ for $n=1,\ldots,20$. Let $n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ be the canonical decomposition of the integer n. By definition, the *radical* of n is the square-free integer $q=\mathrm{rad}(n)=p_1\cdots p_s$ consisting of the product of the p_i's. The computation of the cyclotomic polynomial $\Phi_n(z)$ is greatly simplified by making use of the well-known reduction formula

$$\Phi_n(z)=\Phi_q(z^{n/q}),\quad q=\mathrm{rad}(n). \tag{2.3}$$

This implies that the computation of the exponential generating function of the sequence $(U_n(N))_{N\geq 0}$ is reduced to that of $(U_q(N))_{N\geq 0}$ as follows.

Proposition 2.3 (reduction formula for $U_n(N)$). *Let* $n \geq 1$ *and* $q = \mathrm{rad}(n)$. *Then,*

$$\sum_{N\geq 0} U_n(N)\frac{X^N}{N!} = \left(\sum_{N\geq 0} U_q(N)\frac{X^N}{N!}\right)^{n/q}. \tag{2.4}$$

Proof: Using the remainder function, we have by linearity,

$$R_{\vec{m}}(z) = \mathrm{rem}(P_{\vec{m}}(z), \Phi_n(z)) = \sum_{k=0}^{n-1} m_k \mathrm{rem}(z^k, \Phi_n(z)). \tag{2.5}$$

Now, for $0 \leq \nu \leq q-1$, consider the euclidean division

$$z^\nu = \Phi_q(z)Q_\nu(z) + \rho_\nu(z), \tag{2.6}$$

where $\rho_\nu(z) = \mathrm{rem}(z^\nu, \Phi_q(z))$. The substitution $z \to z^{n/q}$ in (2.6) followed by a multiplication by z^r gives, using (2.3), $z^{\nu n/q+r} = \Phi_q(z^{n/q})z^rQ_\nu(z^{n/q}) + z^r\rho_\nu(z^{n/q}) = \Phi_n(z)z^rQ_\nu(z^{n/q}) + z^r\rho_\nu(z^{n/q})$. Let $k = \nu n/q + r$, where $0 \leq r < n/q$. Then,

$$\deg z^r\rho_\nu(z^{n/q}) = r + \frac{n}{q}\deg\rho_\nu(z) \leq r + \frac{n}{q}(\varphi(q)-1) = r + \varphi(n) - \frac{n}{q} < \varphi(n).$$

This implies that $\mathrm{rem}(z^k, \Phi_n(z)) = z^r\rho_\nu(z^{n/q})$. Substituting this into (2.5) and collecting terms, we find that the $\varphi(n)$ conditions for admissibility, $[l_k(m_0, m_1, \ldots, m_{n-1}) = 0]_{0\leq k\leq\varphi(n)-1}$, split into n/q blocks of $\varphi(q)$ conditions, $[l_i(m_j, m_{\frac{n}{q}+j}, m_{2\frac{n}{q}+j}, \ldots, m_{(q-1)\frac{n}{q}+j}) = 0]_{0\leq i\leq\varphi(q)-1}, \quad 0 \leq j \leq \frac{n}{q}-1$, from which (2.4) follows. □

Table 3 gives the numerical values of $U_n(N)$ for $1 \leq n \leq 20$ and $0 \leq N \leq 20$.

3 Analysis of the sequences

Let us say that a path is *normalized* if its first step is the complex number 1 (i.e. the path starts *horizontally* along the positive real axis). Each normalized path $[1, \omega_n^{\nu_2}, \ldots, \omega_n^{\nu_N}]$ generates, by rotation, n distinct paths $\omega_n^k[1, \omega_n^{\nu_2}, \ldots, \omega_n^{\nu_N}] = [\omega_n^k, \omega_n^{k+\nu_2}, \ldots, \omega_n^{k+\nu_N}]$, $k = 0, 1, \ldots, n-1$. This implies that n divides $U_n(N)$ for every $n \geq 1$ and $N \geq 1$. As Tables 1 and 2 indicate, the structure of the sequence $(U_n(N))_{N\geq 0}$ heavily depend on the arithmetical nature of n. For example, let $n = p$ be a prime number. Then for such values of n, admissibility for a vector $\vec{m} \in \mathbb{N}^p$ means that $m_0 = m_1 = \cdots = m_{p-1}$ since, in this case, $\Phi_p(z) = 1 + z + \cdots + z^{p-1}$ and $R_{\vec{m}}(z) = (m_0 - m_{p-1}) + (m_1 - m_{p-1})z + \cdots + (m_{p-2} - m_{p-1})z^{p-2}$, (see Table 1, for example). Formula (1.2) then takes the form

$$U_p(N) = \frac{N!}{(\frac{N}{p})!^p} \quad \textit{if } p|N, \ 0 \ \textit{otherwise}. \tag{3.1}$$

Note that when $p = 2$, (3.1) corresponds to the classical central binomial coefficients enumerating one-dimensional closed lattice paths of length N. When $p = 3$, (3.1) corresponds to the De Bruijn numbers (sequence A006480 in Sloane-Plouffe encyclopedia [Sloane(2010)]). For prime powers $n = p^\alpha$, we have by Proposition 2.3,

$$\sum_{N\geq 0} U_{p^\alpha}(N)\frac{X^N}{N!} = \left(\sum_{k\geq 0}\frac{X^{kp}}{k!^p}\right)^{p^{\alpha-1}} \tag{3.2}$$

since, in this case $q = p$. Note that when $n = 8 = 2^3$, then $U_8(N)$ is the number of 4-dimensional closed lattice paths in $\mathbb{Z}^4$ of length N starting at the origin (see sequence A039699 in Sloane). The reader can check that, more generally, $U_{2^\alpha}(N)$ is the number of closed lattice paths in $\mathbb{Z}^{2^{\alpha-1}}$ of length N starting at the origin. Interestingly enough, for any other dimension $d \neq 2^{\alpha-1}$, such a connection betweens lattice paths in $\mathbb{Z}^d$ and plane paths whose steps are roots of unity does not exist.

When n is not a prime power, the situation is more delicate. For example, if $n = 6$, then, using the Maple package GFUN [Salvy and Zimmermann(1994)], it can be seen that $(U_n(N))_{N\geq 0}$ satisfies the following linear recurrence with polynomial coefficients,

$$(N+3)^2U_6(N+3) = (N+2)(N+3)U_6(N+2)+24(N+2)^2U_6(N+1)+36(N+1)(N+2)U_6(N) \tag{3.3}$$

with initial conditions $U_6(0) = 1, U_6(1) = 0, U_6(2) = 6$. Such sequences are called polynomially recursive (P-recursive for short) and are characterized by the fact that their (ordinary or exponential) generating series are D-finite (i.e. satisfy a linear differential equation with polynomial coefficients). As a consequence, P-recursive sequences are closed under many operations including linear combinations, pointwise and Cauchy products [Stanley(1980)]. Moreover their asymptotic estimates, as $N \to \infty$, are well behaved. In our context, the general situation is summarized by Theorem 3.2. below. We need first the following technical lemma.

Lemma 3.1 *Let $\vec{t} = (t_1, \ldots, t_{\varphi(n)}) \in \mathbb{C}^{\varphi(n)}$. Then the Laurent polynomial Λ_n satisfies*

$$\max_{\substack{|t_\nu|=1\\ 1\leq\nu\leq\varphi(n)}} |\Lambda_n(\vec{t})| = n. \tag{3.4}$$

Moreover, if $n = p^\alpha$, a prime power, then the maximum value (3.4) *is attained precisely at the p distinct points $(e^{2\pi i\nu/p}, \ldots, e^{2\pi i\nu/p})$, $\nu = 0, \ldots, p-1$ and we have $\Lambda_n(e^{2\pi i\nu/p}, \ldots, e^{2\pi i\nu/p}) = ne^{2\pi i\nu/p}$. If n is not a prime power, then the maximum value* (3.4) *is attained only at the point $(1, \ldots, 1)$ and we have $\Lambda_n(1, \ldots, 1) = n$.*

Proof: By Theorem 2.2, Λ_n can be written as a sum of n terms,

$$\Lambda_n(\vec{t}) = t_1 + \cdots + t_{\varphi(n)} + \Gamma_n(\vec{t}), \tag{3.5}$$

where Γ_n is a sum of $n - \varphi(n)$ unitary Laurent monomials in $t_1, \ldots, t_{\varphi(n)}$. Each of the n terms in Λ_n has modulus 1 when $|t_\nu| = 1$, $\nu = 1, \ldots, \varphi(n)$. Hence (3.4) follows from the triangular inequality and the fact that $\Lambda_n(1, \ldots, 1) = n$. Note that the maximum value in (3.4) is attained only at points $\vec{t^*} = (t_1^*, \ldots, t_{\varphi(n)}^*)$ for which the n monomials take a common value, $e^{i\theta^*}$, say. In particular, from (3.5), we must have $t_1^* = t_2^* = \cdots = t_{\varphi(n)}^* = e^{i\theta^*}$. We consider two cases:

(i) if $n = p^\alpha$, then it can be checked that each term in Γ_n has total degree $-(p-1)$. This implies that $e^{i\theta^*} = e^{-i(p-1)\theta^*}$. That is, $e^{i\theta^*}$ is a p-th root of unity: $e^{2\pi i\nu/p}$, $\nu = 0, \ldots, p-1$;

(ii) if $n \neq p^\alpha$, the situation is more delicate. If we can show that at least one of the terms in Γ_n has total degree 0, then the maximal value in (3.4) will be attained only at the point $(1,\ldots,1)$, since this would imply that $e^{i\theta^*} = (e^{i\theta^*})^0 = 1$. The existence of such a 0-degree term is proved as follows. By (2.2), the general term $t_1^{c_{0,j}} t_2^{c_{1,j}} \cdots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}$ has total degree $\sum_{k=0}^{\varphi(n)-1} c_{k,j}$. When $j = \varphi(n)$, this total degree is 0. To see this, note that $\sum_{k=0}^{\varphi(n)-1} c_{k,j} z^k = \mathrm{rem}(z^j, \Phi_n(z))$. Taking $j = \varphi(n)$, $z = 1$, this gives $\sum_{k=0}^{\varphi(n)-1} c_{k,\varphi(n)} = \mathrm{rem}(z^{\varphi(n)}, \Phi_n(z))|_{z=1} = (z^{\varphi(n)} - \Phi_n(z))|_{z=1} = 0$, since $\Phi_n(1) = 1$ when $n \neq p^\alpha$.

□

Theorem 3.2 *For any $n > 1$, we have an asymptotic estimate of the form*

$$U_n(N) \sim a_n \frac{n^N}{N^{\frac{1}{2}\varphi(n)}} \left(1 + \frac{b_{1,n}}{N} + \frac{b_{2,n}}{N^2} + \ldots\right), \quad \text{as } N \to \infty, \tag{3.6}$$

where a_n, $b_{j,n}$ are independent of N. When $n = p^\alpha$ is a prime power, then N must be a multiple of p as it goes to infinity in (3.6)*. More explicitly, the leading coefficient a_n is given by*

$$a_n = \begin{cases} (n/2\pi)^{\frac{1}{2}\varphi(n)} / \sqrt{\prod_{p|n} p^{\varphi(n)/(p-1)}} & \text{if } n \text{ is not a prime power,} \\ p \cdot (n/2\pi)^{\frac{1}{2}\varphi(n)} / \sqrt{\prod_{p|n} p^{\varphi(n)/(p-1)}} & \text{if } n = p^\alpha \text{ is a prime power.} \end{cases}$$

For each $n \geq 1$, the sequence $(U_n(N))_{N\geq 0}$ is P-recursive but is not algebraic when $n > 2$.

Proof: In order to establish the asymptotic estimate (3.6), first note that the constant term extraction $U_n(N) = \mathrm{CT}(\Lambda_n(t_1,\ldots,t_{\varphi(n)})^N)$ can be expressed as the multiple integral

$$U_n(N) = \frac{1}{(2\pi)^{\varphi(n)}} \int \cdots \int_{(-\pi,\pi]^{\varphi(n)}} \Lambda_n(e^{iu_1},\ldots,e^{iu_{\varphi(n)}})^N du_1 \cdots du_{\varphi(n)} \tag{3.7}$$

which is the average value of Λ_n^N over the $\varphi(n)$-dimensional torus $\{(t_1,\ldots,t_{\varphi(n)}) \in \mathbb{C}^{\varphi(n)} \mid |t_\nu| = 1, \nu = 1,\ldots,\varphi(n)\}$. Now by Theorem 2.2,

$$L_n(\vec{u}) := \Lambda_n(e^{iu_1},\ldots,e^{iu_{\varphi(n)}}) = \sum_{j=0}^{n-1} e^{i\lambda_j(\vec{u})}, \tag{3.8}$$

where $\lambda_j(\vec{u}) = \sum_{k=0}^{\varphi(n)-1} c_{k,j} u_{k+1}$ is a real-valued linear combination of $u_1,\ldots,u_k$, $0 \leq j \leq \varphi(n)-1$. By the triangular inequality, $|L_n(\vec{u})| \leq n$ for every $\vec{u} \in (-\pi,\pi]^{\varphi(n)}$. To obtain the asymptotic estimate of (3.6) it suffices to approximate (3.7) by a gaussian distribution around each point $\vec{u^*} = (u_1^*,\ldots,u_{\varphi(n)}^*) \in (-\pi,\pi]^{\varphi(n)}$ for which the maximum value $|L_n(\vec{u^*})| = |ne^{i\theta^*}| = n$ is attained. This is Laplace's method [De Bruijn(1981)]. By Lemma 3.1,

(i) if $n \neq p^\alpha$, then $\vec{u^*} = \vec{0}$ is the only point in $(-\pi,\pi]^{\varphi(n)}$ for which $|L_n(\vec{u^*})| = n$. In fact $\theta^* = 0$;

(ii) if $n = p^\alpha$, then there are exactly p possible values of u^* for which $|L_n(\vec{u^*})| = n$. In fact $\theta^* = 2\nu\pi/p \mod 2\pi \in (-\pi,\pi], \nu = 0,\ldots,p-1$.

We conclude by estimating (3.7) by a sum of moments of gaussian distributions in the following way:

$$U_n(N) \sim \frac{n^N}{(2\pi)^{\varphi(n)}} \sum_{L_n(\vec{u^*})=ne^{i\theta^*}} e^{iN\theta^*} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{N}{2n}Q^*(\vec{u}-\vec{u^*})} H^*(\vec{u}-\vec{u^*})^N du_1 \dots du_{\varphi(n)},$$

where, for each $\vec{u^*}$ such that $L_n(\vec{u^*}) = ne^{i\theta^*}$,

$$\frac{1}{n}L_n(\vec{u}) = e^{i\theta^*}\left(1 - \frac{1}{2n}Q^*(\vec{u}-\vec{u^*}) + O(\| \vec{u}-\vec{u^*} \|^3)\right) = e^{i\theta^*}e^{-\frac{1}{2n}Q^*(\vec{u}-\vec{u^*})}H^*(\vec{u}-\vec{u^*}), \quad (3.9)$$

where $Q^*(\vec{v})$ is the positive definite quadratic form associated to the symmetric $\varphi(n) \times \varphi(n)$ matrix $K = CC^T$ in which $C = [c_{k,j}]_{0\le k\le\varphi(n)-1, 0\le j\le n-1}$, where the $c_{k,j}$'s are defined by (2.2) and $H^*(\vec{v}) = 1 + O(\| \vec{v} \|^3)$. It turns out that $\det(K) = \prod_{p|n} p^{\varphi(n)/(p-1)}$, which is a consequence of the known fact that the absolute value of the discriminant of $\Phi_n(z)$ is equal to $n^{\varphi(n)} \prod_{p|n} p^{\varphi(n)/(p-1)}$, for $n > 2$.

The P-recursivity of $(U_n(N))_{N\ge0}$ is established as follows. Fix $n \ge 1$ and let $k = \varphi(n)$. We shall show that the series

$$\sum_{N\ge0} U_n(N)X^N = \mathrm{CT}_{t_1,\dots,t_k} \frac{1}{1 - X\Lambda_n(t_1,\dots,t_k)} \quad (3.10)$$

is D-finite in X where $\mathrm{CT}_{t_1,\dots,t_k}$ means constant term extraction relative to the variables $t_1,\dots,t_k$. First, fix integers $m_1 > 0,\dots,m_k > 0$ in such a way that $t_1^{m_1}\dots t_k^{m_k}\Lambda_n(t_1,\dots,t_k)$ is a polynomial in $t_1,\dots,t_k$. The rational function

$$f(t_1,\dots,t_k,X) = \frac{1}{1 - t_1^{m_1}\dots t_k^{m_k}X\Lambda_n(t_1,\dots,t_k)} = \sum_{n_1,\dots,n_k,N\ge0} a(n_1,\dots,n_k,N)t_1^{n_1}\dots t_k^{n_k}X^N \quad (3.11)$$

is obviously D-finite in the variables $t_1,\dots,t_k,X$. By Theorem 2.2, the numbers $U_n(N)$ can be expressed as the following coefficient extraction in $f(t_1,\dots,t_k,X)$:

$$U_n(N) = [t_1^{m_1N}\dots t_k^{m_kN}X^N]f(t_1,\dots,t_k,X).$$

Hence, by (3.10),

$$\sum_{N\ge0} U_n(N)X^N = \sum_{N\ge0} a(m_1N,\dots,m_kN,N)X^N. \quad (3.12)$$

Consider now the algebraic, hence D-finite, series

$$g(t_1,\dots,t_k,X) = \sum_{n_1,\dots,n_k,N\ge0} b(n_1,\dots,n_k,N)t_1^{n_1}\dots t_k^{n_k}X^N,$$

where $b(n_1,\dots,n_k,N) = a(m_1n_1,\dots,m_kn_k,N)$. Formula (3.12) shows that

$$\sum_{N\ge0} U_n(N)X^N = \sum_{N\ge0} b(N,\dots,N,N)X^N$$

which is a (full) diagonal of $g(t_1,\dots,t_k,X)$. We conclude using the fact that any diagonal of a D-finite series is also D-finite, a result due to Lipshitz [Lipshitz(1988)]. The non algebraicity of $(U_n(N))_{N\ge0}$, for each $n > 2$, follows from the fact that $\varphi(n)$ is even and the dominant term of the asymptotic formula contains $N^{-\text{positive integer}}$. This is incoherent with Puiseux expansion around an algebraic singularity. $\square$

A better control of the coefficients $b_{j,n}$ can be achieved by a smooth local change of variables, $\vec{u} = \vec{u^*} + \vec{g}(\vec{w})$, $\vec{g}(\vec{0}) = \vec{0}$ in (3.9) such that $\frac{1}{n}L_n(\vec{u}) = e^{i\theta^*}e^{-\frac{1}{2n}Q^*(\vec{w})}$. This is always possible by Morse Lemma [Morse(1925)]. The first terms of the asymptotic estimates of Theorem 3.2 are given in Table 4 for $n = 1, \dots, 20$.

Corollary 3.3 *If n is not a prime power, then $\exists N_0 = N_0(n)$ such that $U_n(N) > 0$ for $N \geq N_0$.* $\square$

The sequences $(U_n(N))_{N\geq 0}$, $n = 1, 2, \dots$, can be considered in a *dual* way: for each fixed N, one can consider the sequence $(U_n(N))_{n\geq 1}$ by reading each column of Table 3. The first five of these dual sequences, $(U_n(0))_{n\geq 1}, (U_n(1))_{n\geq 1}, \dots, (U_n(4))_{n\geq 1}$, are P-recursive. The fifth one, $(U_n(4))_{n\geq 1}$, can be described as follows: $U_n(4) = 3n(n-1)\chi(2|n)$, where $\chi(T(n)) = 1$ if $T(n)$ is true and 0 otherwise. This can be checked by noting that closed paths of length 4 whose steps are n^{th} roots of unity are (possibly degenerated and non-convex) rhombuses. Following extensive computations we conjecture that $(U_n(5))_{n\geq 1}$ is also P-recursive and is of the form $U_n(5) = 24n\chi(5|n) + 20n(n-3)\chi(6|n)$. We leave open the problem of determining the values of N for which $(U_n(N))_{n\geq 1}$ is P-recursive.

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n	*Linear combinations for admissibility*
1	$[m_0]$
2	$[m_0 - m_1]$
3	$[m_0 - m_2, m_1 - m_2]$
4	$[m_0 - m_2, m_1 - m_3]$
5	$[m_0 - m_4, m_1 - m_4, m_2 - m_4, m_3 - m_4]$
6	$[m_0 - m_2 - m_3 + m_5, m_1 + m_2 - m_4 - m_5]$
7	$[m_0 - m_6, m_1 - m_6, m_2 - m_6, m_3 - m_6, m_4 - m_6, m_5 - m_6]$
8	$[m_0 - m_4, m_1 - m_5, m_2 - m_6, m_3 - m_7]$
9	$[m_0 - m_6, m_1 - m_7, m_2 - m_8, m_3 - m_6, m_4 - m_7, m_5 - m_8]$
10	$[m_0 - m_4 - m_5 + m_9, m_1 + m_4 - m_6 - m_9, m_2 - m_4 - m_7 + m_9, m_3 + m_4 - m_8 - m_9]$
11	$[m_0 - m_{10}, m_1 - m_{10}, m_2 - m_{10}, m_3 - m_{10}, m_4 - m_{10}, m_5 - m_{10}, m_6 - m_{10}, m_7 - m_{10},$ $m_8 - m_{10}, m_9 - m_{10}]$
12	$[m_0 - m_4 - m_6 + m_{10}, m_1 - m_5 - m_7 + m_{11}, m_2 + m_4 - m_8 - m_{10}, m_3 + m_5 - m_9 - m_{11}]$
13	$[m_0 - m_{12}, m_1 - m_{12}, m_2 - m_{12}, m_3 - m_{12}, m_4 - m_{12}, m_5 - m_{12}, m_6 - m_{12}, m_7 - m_{12},$ $m_8 - m_{12}, m_9 - m_{12}, m_{10} - m_{12}, m_{11} - m_{12}]$
14	$[m_0 - m_6 - m_7 + m_{13}, m_1 + m_6 - m_8 - m_{13}, m_2 - m_6 - m_9 + m_{13}, m_3 + m_6 - m_{10} - m_{13},$ $m_4 - m_6 - m_{11} + m_{13}, m_5 + m_6 - m_{12} - m_{13}]$
15	$[m_0 - m_8 - m_9 - m_{10} + m_{13} + m_{14}, m_1 + m_8 - m_{11} - m_{13}, m_2 + m_9 - m_{12} - m_{14},$ $m_3 - m_8 - m_9 + m_{14}, m_4 + m_8 - m_{13} - m_{14}, m_5 - m_8 - m_{10} + m_{13},$ $m_6 - m_9 - m_{11} + m_{14}, m_7 + m_8 + m_9 - m_{12} - m_{13} - m_{14}]$
16	$[m_0 - m_8, m_1 - m_9, m_2 - m_{10}, m_3 - m_{11}, m_4 - m_{12}, m_5 - m_{13}, m_6 - m_{14}, m_7 - m_{15}]$
17	$[m_0 - m_{16}, m_1 - m_{16}, m_2 - m_{16}, m_3 - m_{16}, m_4 - m_{16}, m_5 - m_{16}, m_6 - m_{16}, m_7 - m_{16},$ $m_8 - m_{16}, m_9 - m_{16}, m_{10} - m_{16}, m_{11} - m_{16}, m_{12} - m_{16}, m_{13} - m_{16}, m_{14} - m_{16}, m_{15} - m_{16}]$
18	$[m_0 - m_6 - m_9 + m_{15}, m_1 - m_7 - m_{10} + m_{16}, m_2 - m_8 - m_{11} + m_{17}, m_3 + m_6 - m_{12} - m_{15},$ $m_4 + m_7 - m_{13} - m_{16}, m_5 + m_8 - m_{14} - m_{17}]$
19	$[m_0 - m_{18}, m_1 - m_{18}, m_2 - m_{18}, m_3 - m_{18}, m_4 - m_{18}, m_5 - m_{18}, m_6 - m_{18}, m_7 - m_{18}, m_8 - m_{18},$ $m_9 - m_{18}, m_{10} - m_{18}, m_{11} - m_{18}, m_{12} - m_{18}, m_{13} - m_{18}, m_{14} - m_{18}, m_{15} - m_{18}, m_{16} - m_{18},$ $m_{17} - m_{18}]$
20	$[m_0 - m_8 - m_{10} + m_{18}, m_1 - m_9 - m_{11} + m_{19}, m_2 + m_8 - m_{12} - m_{18}, m_3 + m_9 - m_{13} - m_{19},$ $m_4 - m_8 - m_{14} + m_{18}, m_5 - m_9 - m_{15} + m_{19}, m_6 + m_8 - m_{16} - m_{18}, m_7 + m_9 - m_{17} - m_{19}]$

Tab. 1: The linear combinations $(l_k(\vec{m}))_{0 \le k \le \varphi(n)-1}$ for admissibility, $n = 1, \ldots, 20$.

n	$\Lambda_n(t_1,\ldots,t_{\varphi(n)})$
1	t_1
2	$\left(t_1+{t_1}^{-1}\right)$
3	$\left(t_1+t_2+\frac{1}{t_1t_2}\right)$
4	$\left(t_1+t_2+{t_1}^{-1}+{t_2}^{-1}\right)$
5	$\left(t_1+t_2+t_3+t_4+\frac{1}{t_1t_2t_3t_4}\right)$
6	$\left(t_1+t_2+\frac{t_1}{t_2}+\frac{t_2}{t_1}+{t_1}^{-1}+{t_2}^{-1}\right)$
7	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{1}{t_1t_2t_3t_4t_5t_6}\right)$
8	$\left(t_1+t_2+t_3+t_4+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}\right)$
9	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{1}{t_1t_4}+\frac{1}{t_2t_5}+\frac{1}{t_3t_6}\right)$
10	$\left(t_1+t_2+t_3+t_4+\frac{t_1t_3}{t_2t_4}+\frac{t_2t_4}{t_1t_3}+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}\right)$
11	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9+t_{10}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}}\right)$
12	$\left(t_1+t_2+t_3+t_4+\frac{t_1}{t_3}+\frac{t_3}{t_1}+\frac{t_2}{t_4}+\frac{t_4}{t_2}+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}\right)$
13	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9+t_{10}+t_{11}+t_{12}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}}\right)$
14	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{t_1t_3t_5}{t_2t_4t_6}+\frac{t_2t_4t_6}{t_1t_3t_5}+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}+{t_5}^{-1}+{t_6}^{-1}\right)$
15	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+\frac{t_1t_4t_7}{t_3t_5t_8}+\frac{t_2t_5t_8}{t_1t_4t_6}+\frac{t_1t_6}{t_2t_5t_8}+\frac{t_3t_8}{t_1t_4t_7}+\frac{1}{t_1t_6}+\frac{1}{t_2t_7}+\frac{1}{t_3t_8}\right)$
16	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}+{t_5}^{-1}+{t_6}^{-1}+{t_7}^{-1}+{t_8}^{-1}\right)$
17	$\left(t_1+t_2+\cdots+t_{16}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15}t_{16}}\right)$
18	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{t_1}{t_4}+\frac{t_4}{t_1}+\frac{t_2}{t_5}+\frac{t_5}{t_2}+\frac{t_3}{t_6}+\frac{t_6}{t_3}+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}+{t_5}^{-1}+{t_6}^{-1}\right)$
19	$\left(t_1+t_2+\cdots+t_{18}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15}t_{16}t_{17}t_{18}}\right)$
20	$\left(t_1+t_2+\cdots+t_8+\frac{t_1t_5}{t_3t_7}+\frac{t_3t_7}{t_1t_5}+\frac{t_2t_6}{t_4t_8}+\frac{t_4t_8}{t_2t_6}+{t_1}^{-1}+{t_2}^{-1}+{t_3}^{-1}+{t_4}^{-1}+{t_5}^{-1}+{t_6}^{-1}+{t_7}^{-1}+{t_8}^{-1}\right)$

Tab. 2: The Laurent polynomials Λ_n for $n=1,\cdots,20$.

n	$U_n(N),\ N = 0, \ldots, 20$
1	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
2	1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, 924, 0, 3432, 0, 12870, 0, 48620, 0, 184756
3	1, 0, 0, 6, 0, 0, 90, 0, 0, 1680, 0, 0, 34650, 0, 0, 756756, 0, 0, 17153136, 0, 0
4	1, 0, 4, 0, 36, 0, 400, 0, 4900, 0, 63504, 0, 853776, 0, 11778624, 0, 165636900, 0, 2363904400, 0, 34134779536
5	1, 0, 0, 0, 0, 120, 0, 0, 0, 0, 113400, 0, 0, 0, 0, 168168000, 0, 0, 0, 0, 305540235000
6	1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, 8676360, 47977776, 266378112, 1488801600, 8355739392, 47104393050, 266482019232, 1512589408044, 8610448069080, 49144928795820
7	1, 0, 0, 0, 0, 0, 0, 5040, 0, 0, 0, 0, 0, 0, 681080400, 0, 0, 0, 0, 0, 0
8	1, 0, 8, 0, 168, 0, 5120, 0, 190120, 0, 7939008, 0, 357713664, 0, 16993726464, 0, 839358285480, 0, 42714450658880, 0, 2225741588095168
9	1, 0, 0, 18, 0, 0, 2430, 0, 0, 640080, 0, 0, 215488350, 0, 0, 84569753268, 0, 0, 36905812607664, 0, 0
10	1, 0, 10, 0, 270, 240, 10900, 25200, 551950, 2116800, 32458860, 169092000, 2120787900, 13427013600 149506414200, 1075081207200, 11143223412750, 87198375264000, 865743970019500, 7171730187336000, 69416724049550020
11	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 39916800, 0, 0, 0, 0, 0, 0, 0, 0, 0
12	1, 0, 12, 24, 396, 2160, 23160, 186480, 1845900, 17213280, 171575712, 1703560320, 17365421304, 178323713568, 1856554560432, 19487791106784, 206411964321420, 2201711191213248, 23642813637773616 255355132936441824, 2772650461148938656
13	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6227020800, 0, 0, 0, 0, 0, 0, 0
14	1, 0, 14, 0, 546, 0, 32900, 10080, 2570050, 2540160, 238935564, 465696000, 25142196156, 76886409600, 2900343069624, 12211317518400, 359067702643650, 1915829643087360, 47006105030584700, 300455419743198720, 6437718469449262996
15	1, 0, 0, 30, 0, 360, 7650, 0, 302400, 4544400, 11226600, 324324000, 4310633250, 24324300000, 437404968000, 5634178329780, 45972927000000, 697866761592000, 8962716395833200, 88725951057744000, 1258898645656852200
16	1, 0, 16, 0, 720, 0, 50560, 0, 4649680, 0, 514031616, 0, 64941883776, 0, 9071319628800, 0, 1369263687414480, 0, 219705672931613440, 0, 37024402443528248320
17	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 355687428096000, 0, 0, 0
18	1, 0, 18, 36, 918, 5400, 82800, 801360, 10907190, 132053040, 1802041668, 24199809480, 340640607384, 4834708246368, 70229958125184, 1032223723667136, 15391538570569590, 231935110984687968, 3531542904056225916, 54244559313713885688, 839979883121036697468
19	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 121645100408832000, 0
20	1, 0, 20, 0, 1140, 480, 102800, 151200, 12310900, 38707200, 1812247920, 9574488000, 313983978000, 2391608419200, 62051403928800, 611744666332800, 13627749414064500, 160896284989440000, 3253345101771050000, 43527416858084016000, 829176006298475046640

Tab. 3: The sequences $(U_n(N))_{0 \le N \le 20}$ for $n = 1, \ldots, 20$.

n	*Asymptotic estimate of* $U_n(N)$ *as* $N \to \infty$	*Extra condition*
1	0	*NIL*
2	$\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{2^N}{\sqrt{N}} \left(1 - \frac{1}{4N} + \frac{1}{32N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
3	$\frac{3\sqrt{3}}{2\pi} \cdot \frac{3^N}{N} \left(1 - \frac{2}{3N} + \frac{2}{9N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 3$
4	$\frac{2}{\pi} \cdot \frac{4^N}{N} \left(1 - \frac{1}{2N} + \frac{1}{8N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
5	$\frac{25\sqrt{5}}{4\pi^2} \cdot \frac{5^N}{N^2} \left(1 - \frac{2}{N} + \frac{2}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 5$
6	$\frac{\sqrt{3}}{2\pi} \cdot \frac{6^N}{N} \left(1 - \frac{1}{2N} + \frac{1}{12N^2} + O(\frac{1}{N^3})\right)$	*NIL*
7	$\frac{343\sqrt{7}}{8\pi^3} \cdot \frac{7^N}{N^3} \left(1 - \frac{4}{N} + \frac{8}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 7$
8	$\frac{8}{\pi^2} \cdot \frac{8^N}{N^2} \left(1 - \frac{1}{N} + \frac{1}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
9	$\frac{243\sqrt{3}}{8\pi^3} \cdot \frac{9^N}{N^3} \left(1 - \frac{3}{N} + \frac{4}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 3$
10	$\frac{5\sqrt{5}}{4\pi^2} \cdot \frac{10^N}{N^2} \left(1 - \frac{1}{N} + \frac{3}{4N^2} + O(\frac{1}{N^3})\right)$	*NIL*
11	$\frac{161051\sqrt{11}}{32\pi^5} \cdot \frac{11^N}{N^5} \left(1 - \frac{10}{N} + \frac{50}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{11}$
12	$\frac{3}{\pi^2} \cdot \frac{12^N}{N^2} \left(1 - \frac{1}{N} + \frac{2}{3N^2} + O(\frac{1}{N^3})\right)$	*NIL*
13	$\frac{4826809\sqrt{13}}{64\pi^6} \cdot \frac{13^N}{N^6} \left(1 - \frac{14}{N} + \frac{98}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{13}$
14	$\frac{49\sqrt{7}}{8\pi^3} \cdot \frac{14^N}{N^3} \left(1 - \frac{3}{2N} + \frac{3}{N^2} + O(\frac{1}{N^3})\right)$	*NIL*
15	$\frac{1125}{16\pi^4} \cdot \frac{15^N}{N^4} \left(1 - \frac{4}{N} + \frac{25}{3N^2} + O(\frac{1}{N^3})\right)$	*NIL*
16	$\frac{512}{\pi^4} \cdot \frac{16^N}{N^4} \left(1 - \frac{2}{N} + \frac{9}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
17	$\frac{6975757441\sqrt{17}}{256\pi^8} \cdot \frac{17^N}{N^8} \left(1 - \frac{24}{N} + \frac{288}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{17}$
18	$\frac{81\sqrt{3}}{8\pi^3} \cdot \frac{18^N}{N^3} \left(1 - \frac{3}{2N} + \frac{5}{2N^2} + O(\frac{1}{N^3})\right)$	*NIL*
19	$\frac{322687697779\sqrt{19}}{512\pi^9} \cdot \frac{19^N}{N^9} \left(1 - \frac{30}{N} + \frac{450}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{19}$
20	$\frac{125}{\pi^4} \cdot \frac{20^N}{N^4} \left(1 - \frac{2}{N} + \frac{7}{N^2} + O(\frac{1}{N^3})\right)$	*NIL*

Tab. 4: Asymptotic estimates of $U_n(N)$ as $N \to \infty$, for $n = 1, \ldots, 20$.