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Partition and composition matrices: two matrix analogues of set partitions

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Abstract. This paper introduces two matrix analogues for set partitions; partition and composition matrices. These two analogues are the natural result of lifting the mapping between ascent sequences and integer matrices given in Dukes & Parviainen (2010).

We prove that partition matrices are in one-to-one correspondence with inversion tables. Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s-diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

We show that composition matrices on the set X are in one-to-one correspondence with $(2+2)$-free posets on X. We show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$-free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$-free posets on $\{1,\ldots,n\}$.

Résumé. Ce papier introduit deux analogues matriciels des partitions d'ensembles: les matrices de composition et de partition. Ces deux analogues sont le produit naturel du relèvement de l'application entre suites de montées et matrices d'entiers introduite dans Dukes & Parviainen (2010).

Nous démontrons que les matrices de partition sont en bijection avec les tables d'inversion, les tables d'inversion croissantes correspondant aux matrices de partition avec une relation d'ordre sur les lignes. Les matrices de partition s-diagonales sont classées en fonction de leurs tables d'inversion. Les matrices de partition bidiagonales sont énumérées par la méthode de matrices de transfert et ont même cardinalité que les permutations triables par deux piles en parallèle.

Nous montrons que les matrices de composition sur l'ensemble X sont en bijection avec les ensembles ordonnés $(2+2)$-libres sur X. Nous prouvons que les paires de suites de montées et de permutations sont en bijection avec les ensembles ordonnés $(2+2)$-libres dont les éléments sont les cycles d'une permutation, et nous utilisons cette relation pour exprimer le nombre d'ensembles ordonnés $(2+2)$-libres sur $\{1,\ldots,n\}$.

Keywords: partition matrix, composition matrix, ascent sequence, inversion table

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1 Introduction

The recent papers [4, 2, 7] showed that four entirely different combinatorial structures – (2+2)-free posets, matchings with no neighbour nestings, a new class of pattern avoiding permutations, and a class of upper-triangular integer matrices – are in one-to-one correspondence with special sequences of non-negative integers called *ascent sequences*.

Building on this, Claesson & Linusson [6] presented a bijection from inversion tables (a superset of ascent sequences) to matchings with no left neighbour nestings (a superset of matchings with no neighbour nestings). Although the bijection of [2] is not recovered when the mapping in [6] is restricted to ascent sequences, we consider the second correspondence to be on a level above the first in a hierarchy.

This paper introduces two matrix analogues for set partitions, which does for the correspondence between ascent sequences and matrices what Claesson and Linusson's paper did for matchings. In fact it goes further by presenting a third layer in this yet-to-be-fully-determined hierarchy. Proofs of the results stated in this extended abstract can be found in [5].

Example 1 *Here is an instance of what we shall call a partition matrix:*

$$A = \begin{bmatrix} \{1,2,3\} & \emptyset & \{5,7,8\} & \{9\} \\ \emptyset & \{4\} & \{6\} & \{11\} \\ \emptyset & \emptyset & \emptyset & \{13\} \\ \emptyset & \emptyset & \emptyset & \{10,12\} \end{bmatrix}.$$

Definition 2 *Let X be a finite subset of $\{1, 2, \dots\}$. A* partition matrix *on X is an upper triangular matrix over the powerset of X satisfying the following properties:*

(i) each column and row contain at least one non-empty set;

(ii) the non-empty sets partition X;

(iii) $\mathrm{col}(i) < \mathrm{col}(j) \implies i < j$,

where $\mathrm{col}(i)$ denotes the column in which i is a member. Let Par_n be the collection of all partition matrices on $[1, n] = \{1, \dots, n\}$.

For instance,

$$\mathrm{Par}_3 = \left\{ \begin{bmatrix} \{1,2,3\} \end{bmatrix}, \begin{bmatrix} \{1,2\} & \emptyset \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{2\} \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{3\} \\ \emptyset & \{2\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2,3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset & \emptyset \\ \emptyset & \{2\} & \emptyset \\ \emptyset & \emptyset & \{3\} \end{bmatrix} \right\}.$$

In Section 2 we present a bijection between Par_n and the set of *inversion tables*

$$\mathcal{I}_n = [0, 0] \times [0, 1] \times \cdots \times [0, n-1], \text{ where } [a, b] = \{i \in \mathbb{Z} : a \leq i \leq b\}.$$

Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s-diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

In Section 3 we define composition matrices to be the matrices that satisfy conditions (i) and (ii) of Definition 2. Composition matrices on X are shown to be in one-to-one correspondence with $(2+2)$-free posets on X.

Finally, in Section 4 we show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$-free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$-free posets on $[1, n]$.

Taking the entry-wise cardinality of the matrices in Par_n one gets the matrices of Dukes and Parviainen [7]. In that sense, we generalize the paper of Dukes and Parviainen in a similar way as Claesson and Linusson [6] generalized the paper of Bousquet-Mélou et al. [2]. We note, however, that if we restrict our attention to those inversion tables that enjoy the property of being an *ascent sequence*, then we do *not* recover the bijection of Dukes and Parviainen.

2 Partition matrices and inversion tables

For w a sequence let $\mathrm{Alph}(w)$ denote the set of distinct entries in w. In other words, if we think of w as a word, then $\mathrm{Alph}(w)$ is the (smallest) alphabet on which w is written. Also, let us write $\{a_1, \dots, a_k\}_<$ for a set whose elements are listed in increasing order, $a_1 < \cdots < a_k$. Given an inversion table $w = (x_1, \dots, x_n) \in \mathcal{I}_n$ with $\mathrm{Alph}(w) = \{y_1, \dots, y_k\}_<$ define the $k \times k$ matrix $A = \Lambda(w) \in \mathrm{Par}_n$ by

$$A_{ij} = \left\{ \ell : \, x_\ell = y_i \text{ and } y_j < \ell \leq y_{j+1} \right\},$$

where we let $y_{k+1} = n$. For example, for $w = (0,0,0,3,0,3,0,0,0,8,3,8) \in \mathcal{I}_{12}$ we have $\mathrm{Alph}(w) = \{0,3,8\}$ and

$$\Lambda(w) = \begin{bmatrix} \{1,2,3\} & \{5,7,8\} & \{9\} \\ \emptyset & \{4,6\} & \{11\} \\ \emptyset & \emptyset & \{10,12\} \end{bmatrix} \in \mathrm{Par}_{12}\,.$$

We now define a map $K : \mathrm{Par}_n \to \mathcal{I}_n$. Given $A \in \mathrm{Par}_n$, for $\ell \in [1, n]$ let $x_\ell = \min(A_{*i}) - 1$ where i is the row containing ℓ and $\min(A_{*i})$ is the smallest entry in column i of A. Define

$$K(A) = (x_1, \dots, x_n).$$

Theorem 3 *The map* $\Lambda : \mathcal{I}_n \to \mathrm{Par}_n$ *is a bijection and* K *is its inverse.*

2.1 Statistics on partition matrices and inversion tables

Given $A \in \mathrm{Par}_n$, let $\mathrm{Min}(A) = \{\min(A_{*j}) : j \in [1, \dim(A)]\}$. For instance, the matrix A in Example 1 has $\mathrm{Min}(A) = \{1,4,5,9\}$. From the definition of Λ the following proposition is apparent.

Proposition 4 *If* $w \in \mathcal{I}_n$, $\mathrm{Alph}(w) = \{y_1, \dots, y_k\}_<$ *and* $A = \Lambda(w)$*, then*

$$\mathrm{Min}(A) = \{y_1 + 1, \dots, y_k + 1\} \textit{ and } \dim(A) = |\mathrm{Alph}(w)|.$$

Corollary 5 *The statistic* $\dim$ *on* Par_n *is Eulerian.*

Let us say that i is a *special descent* of $w = (x_1, \dots, x_n) \in \mathcal{I}_n$ if $x_i > x_{i+1}$ and i does not occur in w. Let $\mathrm{sdes}(w)$ denote the number of special descents of w, so

$$\mathrm{sdes}(w) = |\{\, i : x_i > x_{i+1} \text{ and } x_\ell \neq i \text{ for all } \ell \in [1, n] \,\}|.$$

Claesson and Linusson [6] conjectured that sdes has the same distribution on $\mathcal{I}_n$ as the, so-called, bivincular pattern $p = (231, \{1\}, \{1\})$ has on $\mathfrak{S}_n$. An occurrence of p in a permutation $\pi = a_1 \dots a_n$ is a subword $a_i a_{i+1} a_j$ such that $a_{i+1} > a_i = a_j + 1$. We shall define a statistic on partition matrices that is equidistributed with sdes. Given $A \in \mathrm{Par}_n$ let us say that i is a *column descent* if $i+1$ is in the same column as, and above, i in A. Let $\mathrm{cdes}(A)$ denote the number of column descents in A, so

$$\mathrm{cdes}(A) = |\{\, i : \mathrm{row}(i) > \mathrm{row}(i+1) \text{ and } \mathrm{col}(i) = \mathrm{col}(i+1) \,\}|.$$

Proposition 6 *The special descents of $w \in \mathcal{I}_n$ equal the column descents of $\Lambda(w)$. Consequently, the statistic* sdes *on $\mathcal{I}_n$ has the same distribution as* cdes *on* Par_n.

Let us write Mono_n for the collection of matrices in Par_n which satisfy

(iv) $\mathrm{row}(i) < \mathrm{row}(j) \implies i < j$,

where $\mathrm{row}(i)$ denotes the row in which i is a member. We say that an inversion table $(x_1, \dots, x_n)$ is *non-decreasing* if $x_i \le x_{i+1}$ for all $1 \le i < n$.

Theorem 7 *Under the map $\Lambda : \mathrm{Par}_n \to \mathcal{I}_n$, matrices in Mono_n correspond to non-decreasing inversion tables, and $|\,\mathrm{Mono}_n\,| = \binom{2n}{n}/(n+1)$, the nth Catalan number.*

2.2 s-diagonal partition matrices

Theorem 8 *Let $w = (x_1, \dots, x_n) \in \mathcal{I}_n$, $A = \Lambda(w)$ and $\mathrm{Alph}(w) = \{y_1, \dots, y_k\}_<$. Define $y_{k+1} = n$. The matrix A is s-diagonal if and only if for every $i \in [1, n]$ there exists an $a(i) \in [1, k]$ such that*

$$y_{a(i)} < i \le y_{a(i)+1} \text{ and } x_i \in \{y_{a(i)}, y_{a(i)-1}, \dots, y_{\max(1, a(i)-s+1)}\}.$$

Setting $s = 1$ in the above theorem gives us the class of diagonal matrices. These admit a more explicit description which we will now present.

In computer science, *run-length encoding* is a simple form of data compression in which consecutive data elements (runs) are stored as a single data element and its multiplicity. We shall apply this to inversion tables, but for convenience rather than compression purposes. Let $\mathrm{RLE}(w)$ denote the run-length encoding of the inversion table w. For example,

$$\mathrm{RLE}(0,0,0,0,1,1,0,2,3,3) = (0,4)(1,2)(0,1)(2,1)(3,2).$$

A sequence of positive integers $(u_1, \dots, u_k)$ which sum to n is called an *integer composition* of n and we write this as $(u_1, \dots, u_k) \models n$.

Corollary 9 *The set of diagonal matrices in* Par_n *is the image under* Λ *of*

$$\{\, w \in \mathcal{I}_n : (u_1, \dots, u_k) \models n \text{ and } \mathrm{RLE}(w) = (p_0, u_1) \dots (p_{k-1}, u_k) \,\},$$

where $p_0 = 0$, $p_1 = u_1$, $p_2 = u_1 + u_2$, $p_3 = u_1 + u_2 + u_3$, etc.

Since diagonal matrices are in bijection with integer compositions, the number of diagonal matrices in Par_n is 2^{n-1}. Although the bidiagonal matrices do not admit as compact a description in terms of the corresponding inversion tables, we can still count them using the so-called transfer-matrix method [11, §4.7]. Consider the matrix

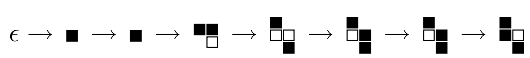
$$B = \begin{bmatrix} \{1,2\} & \{3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{5\} & \emptyset \\ \emptyset & \emptyset & \{4,6\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} \end{bmatrix}.$$

More specifically consider creating B by starting with the empty matrix, ϵ, and inserting the elements 1, . . . , 7 one at a time:

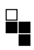
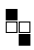
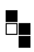
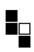
$$\epsilon \to \begin{bmatrix} \{1\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} \\ \emptyset & \emptyset \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{4\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \{5\} \\ \emptyset & \emptyset & \{4\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \{5\} \\ \emptyset & \emptyset & \{4,6\} \end{bmatrix}$$

$$\to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{5\} & \emptyset \\ \emptyset & \emptyset & \{4,6\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} \end{bmatrix}$$

We shall encode (some aspects of) this process like this:



Here, ■ denotes any 1×1 matrix whose only entry is a non-empty set; denotes any 2×2 matrix whose black entries are non-empty;  denotes any matrix of dimension 3 or more, whose entries in the bottom right corner match the picture, that is, the black entries are non-empty; etc. The sequence of pictures does not, in general, uniquely determine a bidiagonal matrix, but each picture contains enough information to tell what pictures can possibly follow it. The matrix below gives all possible transitions (a q records when a new column, and row, is created):

	ϵ													
ϵ	0	q	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	q	q	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	1	q	q	0	0	0	0	0	0	0
	0	0	0	1	1	0	0	0	q	q	0	0	0	0
	0	0	0	0	2	0	0	0	0	0	0	q	q	0
	0	0	0	0	0	$q+1$	q	1	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	q	q	0	0	0	0
	0	0	0	0	0	0	0	2	0	0	0	q	q	0
	0	0	0	0	0	0	0	0	1	0	1	0	0	0
	0	0	0	0	0	0	0	0	q	$q+1$	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	2	q	q	0
	0	0	0	0	0	q	q	0	0	0	0	1	0	1
	0	0	0	0	0	0	0	0	q	q	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	q	q	2

We would like to enumerate paths that start with ϵ and end in a configuration with no empty rows or columns. Letting M denote the above transfer-matrix, this amounts to calculating the first coordinate in

$$(1 - xM)^{-1}[1\,1\,1\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,1]^T.$$

Proposition 10 *We have*

$$\sum_{n\geq 0}\sum_{A\in \mathrm{BiPar}_n} q^{\dim(A)}x^n = \frac{2x^3-(q+5)x^2+(q+4)x-1}{2(q^2+q+1)x^3-(q^2+4q+5)x^2+2(q+2)x-1},$$

where BiPar_n *is the collection of bidiagonal matrices in* Par_n.

We find it interesting that the number of bidiagonal matrices in Par_n is given by the sequence [10, A164870], which corresponds to permutations of $[1,n]$ which are sortable by two pop-stacks in parallel. In terms of pattern avoidance those are the permutations in the class

$$\mathfrak{S}_n(3214, 2143, 24135, 41352, 14352, 13542, 13524).$$

See Atkinson and Sack [1]. A combinatorial proof of this correspondence would be interesting.

Moreover, there are exactly 2^{n-1} permutations of $[1,n]$ which are sortable by one pop-stack; hence equinumerous with the diagonal partition matrices. One might then wonder about permutations which are sortable by three pop-stacks in parallel. Are they equinumerous with tridiagonal partition matrices? Computations show that this is not the case: For $n=6$ there are 646 tridiagonal partition matrices, but only 644 permutations which are sortable by three pop-stacks in parallel.

3 Composition matrices and $(2+2)$-free posets

Consider Definition 2. Define a *composition matrix* to be a matrix that satisfies conditions (i) and (ii), but not necessarily (iii). Let $\mathrm{Comp}_n \supseteq \mathrm{Par}_n$ denote the set of all composition matrices on $[1, n]$. The smallest example of a composition matrix that is not a partition matrix is

$$\begin{bmatrix} \{2\} & \emptyset \\ \emptyset & \{1\} \end{bmatrix}.$$

In this section we shall give a bijection from Comp_n to the set of $(2+2)$-free posets on $[1, n]$. This bijection will factor through a certain union of Cartesian products that we now define. Given a set X, let us write $\binom{X}{x_1,\ldots,x_\ell}$ for the collection of all sequences $(X_1, \ldots, X_\ell)$ that are ordered set partitions of X and $|X_i| = x_i$ for all $i \in [1, \ell]$. For a sequence $(a_1, \ldots, a_i)$ of numbers let

$$\mathrm{asc}(a_1, \ldots, a_i) = |\{j \in [1, i-1] : a_j < a_{j+1}\}|.$$

Following Bousquet-Mélou et al. [2] we define a sequence of non-negative integers $\alpha = (a_1, \ldots, a_n)$ to be an *ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{asc}(a_1, \ldots, a_i)]$ for $0 < i < n$. Let $\mathcal{A}_n$ be the collection of ascent sequences of length n. Define the *run-length record* of α to be the sequence that records the multiplicities of adjacent values in α. We denote it by $\mathrm{RLR}(\alpha)$. In other words, $\mathrm{RLR}(\alpha)$ is the sequence of second coordinates in $\mathrm{RLE}(\alpha)$, the run-length encoding of α. For instance,

$$\mathrm{RLR}(0, 0, 0, 0, 1, 1, 0, 2, 3, 3) = (4, 2, 1, 1, 2).$$

Finally we are in a position to define the set which our bijection from Comp_n to $(2+2)$-free posets on $[1, n]$ will factor through. Let

$$\mathfrak{A}_n = \bigcup_{\alpha \in \mathcal{A}_n} \{\alpha\} \times \binom{[1, n]}{\mathrm{RLR}(\alpha)}.$$

Let $\mathcal{M}_n$ be the collection of upper triangular matrices that contain non-negative integers whose entries sum to n and such that there is no column or row of all zeros. Dukes and Parviainen [7] presented a bijection

$$\Gamma : \mathcal{M}_n \to \mathcal{A}_n.$$

Given $A \in \mathcal{M}_n$, let $\mathrm{nz}(A)$ be the number of non-zero entries in A. Since A may be uniquely constructed, in a step-wise fashion, from the ascent sequence $\Gamma(A)$, we may associate to each non-zero entry A_{ij} its time of creation $T_A(i, j) \in [1, \mathrm{nz}(A)]$. By defining $T_A(i, j) = 0$ if $A_{ij} = 0$ we may view T_A as a $\dim(A) \times \dim(A)$ matrix. Define $\mathrm{Seq}(A) = (y_1, \ldots, y_{\mathrm{nz}(A)})$ where $y_t = A_{ij}$ and $T_A(i, j) = t$.

Example 11 *We have*

$$A = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \implies T_A = \begin{bmatrix} 1 & 0 & 5 & 8 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and $\mathrm{Seq}(A) = (3, 1, 2, 1, 3, 1, 1, 1)$.

The following is a straightforward consequence of the construction rules given by Dukes and Parviainen [7].

Lemma 12 *Given* $A \in \mathcal{M}_n$*, we have that* $\mathrm{Seq}(A) = \mathrm{RLR}(\Gamma(A))$.

For a matrix $A \in \mathrm{Comp}_n$ define $\mathrm{Card}(A)$ as the matrix obtained from A by taking the cardinality of each of its entries. Note that $A \mapsto \mathrm{Card}(A)$ is a surjection from Par_n onto $\mathcal{M}_n$. Define $E(A)$ as the ordered set partition $(X_1, \ldots, X_{\mathrm{nz}(A)})$, where $X_t = A_{ij}$ for $t = T_{\mathrm{Card}(A)}(i,j)$. Finally, define $f : \mathrm{Comp}_n \to \mathfrak{A}_n$ by

$$f(A) = \big(\Gamma(\mathrm{Card}(A)),\, E(A)\big).$$

Example 13 *Let us calculate* $f(A)$ *for*

$$A = \begin{bmatrix} \{3,8\} & \{6\} & \emptyset \\ \emptyset & \{2,5,7\} & \emptyset \\ \emptyset & \emptyset & \{1,4\} \end{bmatrix}.$$

We have

$$\mathrm{Card}(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad T_{\mathrm{Card}(A)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$f(A) = \big(\Gamma(\mathrm{Card}(A)),\, E(A)\big) = \big((0,0,1,1,1,0,2,2),\, \{3,8\}\{2,5,7\}\{6\}\{1,4\}\big).$$

We now define a map $g : \mathfrak{A}_n \to \mathrm{Comp}_n$. For $(w,\chi) \in \mathfrak{A}_n$ with $\chi = (X_1, \ldots, X_k)$ let $g(w,\chi) = A$, where $A_{ij} = X_t$, $t = T_B(i,j)$ and $B = \Gamma^{-1}(w)$. It is easy to verify that $f(\mathrm{Comp}_n) \subseteq \mathfrak{A}_n$, $g(\mathfrak{A}_n) \subseteq \mathrm{Comp}_n$, $g(f(w,\chi)) = (w,\chi)$ for $(w,\chi) \in \mathfrak{A}_n$, and $f(g(A)) = A$ for $A \in \mathrm{Comp}_n$. Thus the following theorem.

Theorem 14 *The map* $f : \mathrm{Comp}_n \to \mathfrak{A}_n$ *is a bijection and* g *is its inverse.*

Next we will give a bijection ϕ from $\mathfrak{A}_n$ to $\mathfrak{P}_n$, the set of $(2+2)$-free posets on $[1,n]$. Let $(\alpha,\chi) \in \mathfrak{A}_n$ with $\chi = (X_1, \ldots, X_\ell)$. Assuming that $X_i = \{x_1, \ldots, x_k\}_<$ define the word $\hat{X}_i = x_1 \ldots x_k$ and let $\hat{\chi} = \hat{X}_1 \ldots \hat{X}_\ell$. From this, $\hat{\chi}$ will be a permutation of the elements $[1,n]$. Let $\hat{\chi}(i)$ be the ith letter of this permutation.

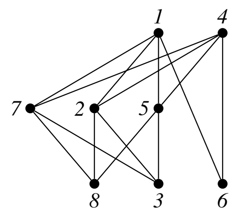
For $(\alpha,\chi) \in \mathfrak{A}_n$ define $\phi(\alpha,\chi)$ as follows: Construct the poset element by element according to the construction rules of [2] on the ascent sequence α. Label with $\hat{\chi}(i)$ the element inserted at step i.

The inverse of this map is also straightforward to state and relies on the following crucial observation [8, Prop. 3] concerning indistinguishable elements in an unlabeled $(2+2)$-free poset. Two elements in a poset are called *indistinguishable* if they obey the same relations relative to all other elements.

Let P be an unlabeled poset that is constructed from the ascent sequence $\alpha = (a_1, \ldots, a_n)$. Let p_i and p_j be the elements that were created during the ith and jth steps of the construction given in [2, Sect. 3]. The elements p_i and p_j are indistinguishable in P if and only if $a_i = a_{i+1} = \cdots = a_j$.

Define $\psi : \mathfrak{P}_n \to \mathfrak{A}_n$ as follows: Given $P \in \mathfrak{P}_n$ let $\psi(P) = (\alpha,\chi)$ where α is the ascent sequence that corresponds to the poset P with its labels removed, and χ is the sequence of sets $(X_1, \ldots, X_m)$ where X_i is the set of labels that corresponds to all the indistinguishable elements of P that were added during the ith run of identical elements in the ascent sequence.

Example 15 *Consider the* $(2+2)$*-free poset*



The unlabeled poset corresponding to P *has ascent sequence* $(0,0,1,1,1,0,2,2)$*. There are four runs in this ascent sequence. The first run of two 0s inserts the elements* 3 *and* 8*, so we have* $X_1 = \{3,8\}$*. Next the run of three 1s inserts elements* 2*,* 5 *and* 7*, so* $X_2 = \{2,5,7\}$*. The next run is a run containing a single* 0*, and the element inserted is* 6*, so* $X_3 = \{6\}$*. The final run of two 2s inserts elements* 1 *and* 4*, so* $X_4 = \{1,4\}$*. Thus we have*

$$\psi(P) = \big((0,0,1,1,1,0,2,2),\ \{3,8\}\{2,5,7\}\{6\}\{1,4\}\big).$$

It is straightforward to check that ϕ and ψ are each others inverses. Consequently, we have the following theorem.

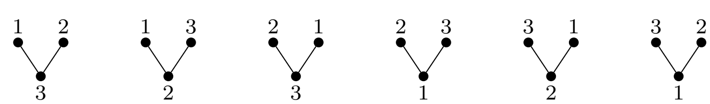
Theorem 16 *The map* $\phi : \mathfrak{A}_n \to \mathfrak{P}_n$ *is a bijection and* ψ *is its inverse.*

Let $\mathcal{C}_n$ be the collection of composition matrices M on $\{1,\ldots,n\}$ with the following property: in every row of M, the entries in the sets are increasing from left to right. For example $(\{1,2\},\emptyset,\{3,4\})$ is a valid row, but $(\{1,3\},\emptyset,\{2,4\})$ is not. Combining the properties of the map Λ from Section 2, Theorem 7, and the observation that relabeling the elements in a partition matrix which correspond to the same element of the inversion table when written from left to right in the rows in increasing order yield the above stated property of composition matrices, we have the following. The enumeration follows from Stanley [12, Exer. 5.49] or Haglund & Loehr [9, §2].

Corollary 17 *Let* $\mathcal{G}_n$ *be the collection of all pairs* (α,χ) *where* α *is a non-decreasing inversion table of length* n*, and* χ *is an ordered set partition of* $\{1,\ldots,n\}$ *where the sequences of sizes of the sets equals* $RLR(\alpha)$*. Then there is a bijection between* $\mathcal{C}_n$ *and* $\mathcal{G}_n$*, and* $|\mathcal{C}_n| = (n+1)^{n-1}$*.*

4 The number of $(2+2)$-free posets on $[1,n]$

Let us consider *plane (2+2)-free posets* on $[1,n]$. That is, $(2+2)$-free posets on $[1,n]$ with a canonical embedding in the plane. For instance, these are six *different* plane $(2+2)$-free posets on $[1,3] = \{1,2,3\}$:



By definition, if u_n is the number of unlabeled $(2+2)$-free posets on n nodes, then $u_n n!$ is the number of plane $(2+2)$-free posets on $[1,n]$. In other words, we may identify the set of plane $(2+2)$-free posets on $[1,n]$ with the Cartesian product $\mathcal{P}_n \times \mathfrak{S}_n$, where $\mathcal{P}_n$ denotes the set of unlabeled $(2+2)$-free posets on n nodes and $\mathfrak{S}_n$ denotes the set of permutations on $[1,n]$. We shall demonstrate the isomorphism

$$\bigcup_{\pi\in\mathfrak{S}_n} \mathfrak{P}(\mathrm{Cyc}(\pi)) \simeq \mathcal{P}_n \times \mathfrak{S}_n, \tag{1}$$

where $\mathrm{Cyc}(\pi)$ is the set of (disjoint) cycles of π and $\mathfrak{P}(\mathrm{Cyc}(\pi))$ is the set of $(2+2)$-free posets on those cycles. As an illustration we consider the case $n = 3$. On the right-hand side we have $|\mathcal{P}_3 \times \mathfrak{S}_3| = |\mathcal{P}_3||\mathfrak{S}_3| = 5\cdot 6 = 30$ plane $(2+2)$-free posets. Taking the cardinality of the left-hand side we get

$$\begin{aligned} &|\mathfrak{P}\{(1),(2),(3)\}| + |\mathfrak{P}\{(1),(23)\}| + |\mathfrak{P}\{(12),(3)\}| + |\mathfrak{P}\{(2),(13)\}| + |\mathfrak{P}\{(123)\}| + |\mathfrak{P}\{(132)\}| \\ &= |\mathfrak{P}_3| + 3|\mathfrak{P}_2| + 2|\mathfrak{P}_1| = 19 + 3\cdot 3 + 2\cdot 1 = 30. \end{aligned}$$

Bousquet-Mélou et al. [2] gave a bijection Ψ from $\mathcal{P}_n$ to $\mathcal{A}_n$, the set of ascent sequences of length n. Recall also that in Theorem 16 we gave a bijection ϕ from $\mathfrak{P}_n$ to $\mathfrak{A}_n$. Of course, there is nothing special about the ground set being $[1,n]$ in Theorem 16; so, for any finite set X, the map ϕ can be seen as a bijection from $(2+2)$-free posets on X to the set

$$\mathfrak{A}(X) = \bigcup_{\alpha\in\mathcal{A}_{|X|}} \{\alpha\} \times \binom{X}{\mathrm{RLR}(\alpha)}.$$

In addition, the fundamental transformation [3] is a bijection between permutations with exactly k cycles and permutations with exactly k left-to-right minima. Putting these observations together it is clear that to show (1) it suffices to show

$$\bigcup_{\pi\in\mathfrak{S}_n} \mathfrak{A}(\mathrm{LMin}(\pi)) \simeq \mathcal{A}_n \times \mathfrak{S}_n, \tag{2}$$

where $\mathrm{LMin}(\pi)$ is the set of segments obtained by breaking π apart at each left-to-right minima. For instance, the left-to-right minima of $\pi = 5731462$ are 5, 3 and 1; so $\mathrm{LMin}(\pi) = \{57, 3, 1462\}$.

Let us now prove (2) by giving a bijection h from the left-hand side to the right-hand side. To this end, fix a permutation $\pi \in \mathfrak{S}_n$ and let $k = |\mathrm{LMin}(\pi)|$ be the number of left-to-right minima in π. Assume that $\alpha = (a_1,\dots,a_k)$ is an ascent sequence in $\mathcal{A}_k$ and that $\chi = (X_1,\dots,X_r)$ is an ordered set partition in $\binom{\mathrm{LMin}(\pi)}{\mathrm{RLR}(\alpha)}$. To specify the bijection h let

$$h(\alpha,\chi) = (\beta,\tau)$$

where $\beta \in \mathcal{A}_n$ and $\tau \in \mathfrak{S}_n$ are defined in the next paragraph.

For each $i \in [1,r]$, first order the blocks of X_i decreasingly with respect to first (and thus minimal) element, then concatenate the blocks to form a word $\hat{X}_i$. Define the permutation τ as the concatenation of the $\hat{X}_i$s:

$$\tau = \hat{X}_1 \dots \hat{X}_k.$$

Let $i_1 = 1$, $i_2 = i_1 + |X_1|$, $i_3 = i_2 + |X_2|$, etc. By definition, these are the indices where the ascent sequence α changes in value. Define β by

$$\mathrm{RLE}(\beta) = (a_{i_1}, x_1)\dots(a_{i_k}, x_k), \text{ where } x_i = |\hat{X}_i|.$$

Consider the permutation $\pi = \mathrm{A9B68D4F32C175E} \in \mathfrak{S}_{15}$ (in hexadecimal notation). Then $\mathrm{LMin}(\pi) = \{\mathrm{A}, \mathrm{9B}, \mathrm{68D}, \mathrm{4F}, 3, \mathrm{2C}, \mathrm{175E}\}$. Assume that

$$\alpha = (0, 0, 1, 2, 2, 2, 0);$$
$$\chi = \{\mathrm{2C}, \mathrm{68D}\}\{\mathrm{9B}\}\{3, \mathrm{175E}, \mathrm{4F}\}\{\mathrm{A}\}.$$

Then we have $\hat{X}_1 = \mathrm{68D2C}$, $\hat{X}_2 = \mathrm{9B}$, $\hat{X}_3 = \mathrm{4F3175E}$ and $\hat{X}_4 = \mathrm{A}$. Also, $i_1 = 1$, $i_2 = 1 + 2 = 3$, $i_3 = 3 + 1 = 4$ and $i_4 = 4 + 3 = 7$. Consequently,

$$\beta = (0, 0, 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 0);$$
$$\tau = 6\ 8\ \mathrm{D}\ 2\ \mathrm{C}\ 9\ \mathrm{B}\ 4\ \mathrm{F}\ 3\ 1\ 7\ 5\ \mathrm{E}\ \mathrm{A}.$$

It is clear how to reverse this procedure: Split τ into segments according to where β changes in value when reading from left to right. With τ as above we get

$$(\mathrm{68D2C}, \mathrm{9B}, \mathrm{4F3175E}, \mathrm{A}) = (\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4)$$

We have thus recovered $\hat{X}_1$, $\hat{X}_2$, etc. Now $X_i = \mathrm{LMin}(\hat{X}_i)$, and we thus know χ. It only remains to recover α. Assume that $\mathrm{RLE}(\beta) = (b_1, x_1) \ldots (b_k, x_k)$, then $\mathrm{RLE}(\alpha) = (b_1, |X_1|) \ldots (b_k, |X_k|)$. This concludes the proof of (2). Let us record this result.

Theorem 18 *The map* $h : \cup_{\pi \in \mathfrak{S}_n} \mathfrak{A}(\mathrm{LMin}(\pi)) \to \mathcal{A}_n \times \mathfrak{S}_n$ *is a bijection.*

As previously explained, (1) also follows from this theorem. Let us now use (1) to derive an exponential generating function $L(t)$ for the number of $(2 + 2)$-free posets in $[1, n]$. Bousquet-Mélou et al. [2] gave the following *ordinary* generating function for *unlabeled* $(2 + 2)$-free posets on n nodes:

$$\begin{aligned} P(t) &= \sum_{n \geq 0} \prod_{i=1}^{n} \left(1 - (1 - t)^i\right) \\ &= 1 + t + 2t^2 + 5t^3 + 15t^4 + 53t^5 + 217t^6 + 1014t^7 + 5335t^8 + O(t^9). \end{aligned}$$

This is, of course, also the exponential generating function for plane $(2 + 2)$-free posets on $[1, n]$. Moreover, the exponential generating function for cyclic permutations is $\log(1/(1 - t))$. On taking the union over $n \geq 0$ of both sides of (1) it follows that $L(\log(1/(1 - t))) = P(t)$; so $L(t) = P(1 - e^{-t})$.

Corollary 19 *The exponential generating function for* $(2 + 2)$*-free posets is*

$$\begin{aligned} L(t) &= \sum_{n \geq 0} \prod_{i=1}^{n} \left(1 - e^{-ti}\right) \\ &= 1 + t + 3\frac{t^2}{2!} + 19\frac{t^3}{3!} + 207\frac{t^4}{4!} + 3451\frac{t^5}{5!} + 81663\frac{t^6}{6!} + 2602699\frac{t^7}{7!} + O(t^8). \end{aligned}$$

This last result also follows from a result of Zagier [13, Eq. 24] and a bijection, due to Bousquet-Mélou et al. [2], between unlabeled $(2 + 2)$-free posets and certain matchings. See also Exercises 14 and 15 in Chapter 3 of the second edition of Enumerative Combinatorics Volume 1 (available on R. Stanley's homepage).

References

[1] M. D. Atkinson and J.-R. Sack, Pop-stacks in parallel, *Information Processing Letters* **70** (1999), 63–67.

[2] M. Bousquet-Mélou, A. Claesson, M. Dukes and S. Kitaev, $(2+2)$-free posets, ascent sequences and pattern avoiding permutations, *Journal of Combinatorial Theory Series A* **117** (2010), no. 7, 884–909.

[3] P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, *Lecture Notes in Mathematics*, No. 85 Springer-Verlag, Berlin-New York 1969.

[4] A. Claesson, M. Dukes and S. Kitaev, A direct encoding of Stoimenow's matchings as ascent sequences, *Australasian Journal of Combinatorics* **49** (2011), 47–59.

[5] A. Claesson, M. Dukes and M. Kubitzke, Partition and composition matrices, *Journal of Combinatorial Theory Series A* **118** (2011), no. 5, 1624–1637.

[6] A. Claesson and S. Linusson, $n!$ matchings, $n!$ posets, *Proceedings of the American Mathematical Society* **139** (2011) 435–449.

[7] M. Dukes and R. Parviainen, Ascent sequences and upper triangular matrices containing non-negative integers, *Electronic Journal of Combinatorics* **17** (2010), no. 1, #53 (16pp).

[8] M. Dukes, S. Kitaev, J. Remmel and E. Steingrímsson, Enumerating $(2+2)$-free posets by indistinguishable elements, *Journal of Combinatorics*, to appear (2011).

[9] J. Haglund and N. Loehr, A Conjectured Combinatorial Formula for the Hilbert Series for Diagonal Harmonics, *Discrete Mathematics* **298** (2005) 189–204.

[10] N. J. A. Sloane, (2009), The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/∼njas/sequences/.

[11] R. P. Stanley, Enumerative Combinatorics Volume 1, Cambridge University Press, 1997.

[12] R. P. Stanley, Enumerative Combinatorics Volume 2, Cambridge University Press, 2001.

[13] D. Zagier, Vassiliev invariants and a strange identity related to the Dedeking eta-function, *Topology* **40** (2001), 945–960.