Partially Marked Pattern Families

Mark Tiefenbruck

June 14, 2012

Matchings

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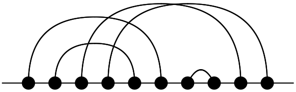

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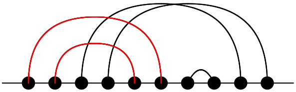

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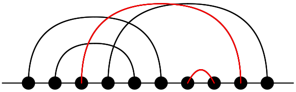

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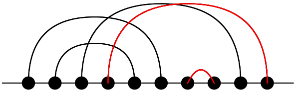

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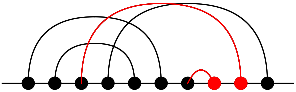

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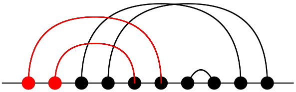

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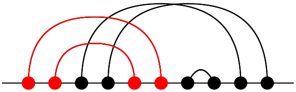

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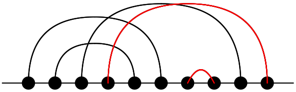

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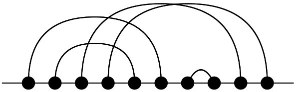

- It has nestings.
- Some are right-nestings.
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- Some are neither.
- Let NRN_n be the set of matchings with n arcs and no right-nestings.

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- $\sigma_i, \sigma_{i+1}, \sigma_i + 1$ form a *p*-pattern if $\sigma_i > \sigma_{i+1}$ and $\sigma_i + 1$ is to the right of σ_i.

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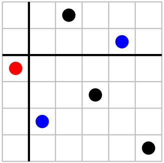

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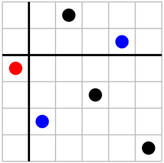

- Claesson and Linusson conjectured that left-nestings in NRN_n and p-patterns in S_n have the same distribution.

First Try

- Build permutations by repeatedly inserting new largest number:

$$1$$

First Try

- Build permutations by repeatedly inserting new largest number:

21

First Try

- Build permutations by repeatedly inserting new largest number:

$$23\color{red}{3}1$$

First Try

- Build permutations by repeatedly inserting new largest number:

$$4231$$

First Try

- Build permutations by repeatedly inserting new largest number:

42351

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- When do we create a new p-pattern?

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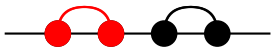

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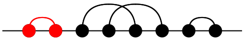

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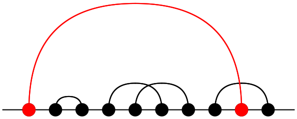

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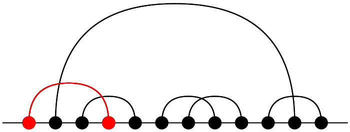

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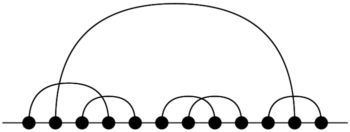

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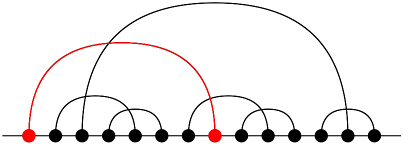

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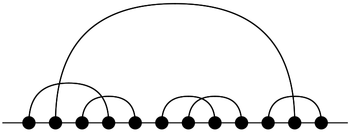

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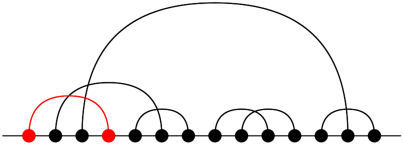

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What Goes Wrong

- Consider the permutation 25314:

$1 \rightarrow$

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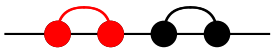

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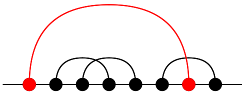

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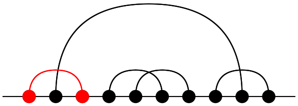
25314 $\rightarrow$ 

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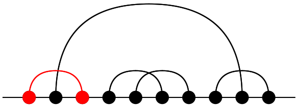
25314 $\rightarrow$ 

- 314 is a *p*-pattern, but the corresponding left-nesting is broken.

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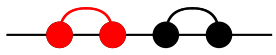

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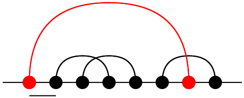

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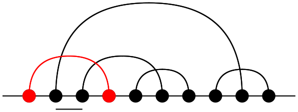

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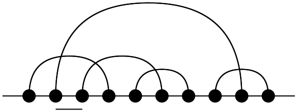

- The image of 23514 still has a left-nesting, so we're not done.

Actually....

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Obvious Corollary

$$\sum_{\substack{\text{partially}\\ \text{marked}\\ \text{object1s}}} x^{\#\text{ of marked patterns}} = \sum_{\substack{\text{partially}\\ \text{marked}\\ \text{object2s}}} x^{\#\text{ of marked patterns}}$$

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- For example, in (27498), 982 is a 321-cycle-match.
- Let $\tau_{\mathrm{cyc}}(\sigma)$ be the number of τ-cycle-matches in σ's cycles.
- If τ begins with 1, Jones and Remmel showed that $|S_n(\tau_{\mathrm{cyc}})| = |S_n(\tau\text{-mch})|$.

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- For example, $(\underline{142}\overline{4253}\underline{3})$ is covered by 3142-cycle-matches.
- Jones conjectured that if τ cannot cover a cycle, then $|S_n(\tau_{\mathrm{cyc}})| = |S_n(\tau\text{-mch})|$.

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- This actually proves τ_{cyc} and τ-mch have the same distribution.

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- For $\sigma = (143)(2)$, $\sigma \to 2143$.

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- Done!

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- For example, $|S_n(1234, 2134)| = |S_n(3412, 3421)|$, but the theorem tells us more.

The End

Thanks for listening!