Some Adin-Roichman-Mansour type identities

Sen-Peng Eu\textsuperscript{1,2} \quad Chien-Tai Ting\textsuperscript{1,2}

\begin{itemize}
\item \textsuperscript{1}National University of Kaohsiung, Taiwan, ROC
\item \textsuperscript{2}Air Force Academy, Taiwan, ROC
\end{itemize}

\textsuperscript{1}國立高雄大學, \textsuperscript{2}空軍軍官學校
Outline of this talk

- Identities of Ardin-Roichman & Mansour
- Some ARM identities
  - $\text{Alt}_n(321)$
  - $\text{Bax}_n(123)$
  - $\text{DS}_n(312)$
- Idea of proof
- Problems & Discussions

- Part of these works are joint with Fu, Pan, Yan.
Part 1

Adin-Roichman-Mansour type identities
Pattern avoiding permutations

• $\pi$ is $\sigma$-avoiding: (we know that)...
  
  25134 is 321-avoiding.

  • Let $\mathfrak{S}_n(321)$ be the 321-avoiding permutations in $\mathfrak{S}_n$.

Theorem [Simion, Schmidt, 1985]

$$\sum_{\pi \in \mathfrak{S}_n(321)} 1 = c_n,$$

where $c_n = \frac{1}{n+1} \binom{2n}{n}$ is the Catalan number.
Sign-balance

• Amazingly, the signed counting is also a Catalan numbers.

**Theorem** [Simion, Schmidt, 1985]

\[
\sum_{\pi \in \mathcal{S}_n(321)} (-1)^{\text{inv}(\pi)} = \begin{cases} 
\frac{C_{n-1}}{2}, \\
0,
\end{cases}
\]

i.e.,

**Theorem** [Simion, Schmidt, 1985]

\[
\begin{array}{l}
\left\{ \begin{array}{l}
\sum_{\pi \in \mathcal{S}_{2n+1}(321)} (-1)^{\text{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n(321)} 1.
\end{array} \right.
\end{array}
\]

\[
\begin{array}{l}
\left\{ \begin{array}{l}
\sum_{\pi \in \mathcal{S}_{2n}(321)} (-1)^{\text{inv}(\pi)} = 0.
\end{array} \right.
\end{array}
\]
Adin-Roichman identities

• In 2004, Adin and Roichman gave a refinement.

\[ \text{ldes}(\pi) := \text{last descent of } \pi \]
\[ = \max\{1 \leq i \leq n - 1 : \pi(i) > \pi(i + 1)\} \]
\[ (\text{ldes}(\pi) := 0 \text{ if } \pi = \text{id}) \]

\[ \cdot \text{ldes}(216534) = 4. \]
Adin-Roichman’s identities

**Theorem** [Adin, Roichman, 2004, SLC]

\[
\begin{align*}
\sum_{\pi \in \mathcal{S}_{2n+1}(321)} (-1)^{\text{inv}(\pi)} q^{\text{ldes}(\pi)} &= \sum_{\pi \in \mathcal{S}_{n}(321)} q^{2\text{ldes}(\pi)} \quad (n \geq 0). \\
\sum_{\pi \in \mathcal{S}_{2n}(321)} (-1)^{\text{inv}(\pi)} q^{\text{ldes}\pi} &= (1 - q) \sum_{\pi \in \mathcal{S}_{n}(321)} q^{2\text{ldes}(\pi)} \quad (n \geq 1).
\end{align*}
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum(-1)^{\text{inv}(\pi)} q^{\text{ldes}(\pi)}$</td>
<td>1</td>
<td>1 - $q$</td>
<td>1</td>
<td>$(1 - q)(1 + q^2)$</td>
<td>1 + $q^2$</td>
</tr>
<tr>
<td>$\sum q^{\text{ldes}(\pi)}$</td>
<td>1</td>
<td>1 + $q$</td>
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When $q = 1$, they reduces to Simon-Schmidt’s identities.
Mansour’s identities

At the same time, Mansour consider the $\mathfrak{S}_n(132)$.

\[
\text{find}(\pi) := \text{the index of the letter ‘1’ in } \pi = \pi^{-1}(1)
\]

\[
\text{find}(216534) = 2.
\]

**Theorem** [Mansour, 2004, SLC] For $n \geq 1$,

\[
\begin{align*}
\sum_{\pi \in \mathfrak{S}_{2n+1}(132)} (-1)^{\text{inv}(\pi)} q^{\text{ldes}(\pi)} &= \sum_{\pi \in \mathfrak{S}_{2n+1}(132)} q^{\text{find}(\pi)-1}. \\
\sum_{\pi \in \mathfrak{S}_{2n}(132)} (-1)^{\text{inv}(\pi)} q^{\text{ldes}\pi} &= (1 - q) \sum_{\pi \in \mathfrak{S}_n(132)} q^{2(\text{find}(\pi)-1)}.
\end{align*}
\]
2n reduced to n phenomenon

• The Adin-Roichman-Mansour identities are essentially
  “2n reduces to n phenomena”
  • “Signed enumerator on size 2n” = “enumerator on size n”.

• “ARM-type identities"
  • bending the ARM — folding in half and double in thickness!

• In this talk, we present some new instances.
Part 2

Alternating permutations (321)
Alternating permutations

- $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is alternating if
  \[ \pi_1 > \pi_2 < \pi_3 > \pi_4 < \ldots \]

- $\text{Alt}_n :=$ set of Alternating permutations of length $n$.
  \[ \sum_{n \geq 0} |\text{Alt}_n| \frac{x^n}{n!} = \tan x + \sec x. \]
  \[ |\text{Alt}_n|_{n \geq 0} = 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots \]

- $\text{Alt}_n(321) := \text{Alt}_n$ and avoiding 321.

**Theorem** [Deutsch, Reifegerste 2003, Mansour 2003]

\[ |\text{Alt}_{2n}(321)| = |\text{Alt}_{2n-1}(321)| = \frac{1}{n+1} \binom{2n}{n} \]

- $|\text{Alt}_6(321)| = 5. (214365, 215364, 314265, 315264, 415263)$. 
- $|\text{Alt}_5(321)| = 5. (21435, 21435, 31425, 31524, 41523)$. 

Signed counting

- Motivation: signed counting of $\text{Alt}_{2n}(321)$

<table>
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<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>5</th>
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<th>9</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
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</thead>
<tbody>
<tr>
<td>$\sum (-1)^{\text{inv}}$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>-5</td>
<td>5</td>
</tr>
</tbody>
</table>

- Theorem [Pan, Ting, 2011]

\[
\sum_{\text{Alt}_{4n+2}(321)} (-1)^{\text{inv}(\pi)} = (-1)^{n+1} \sum_{\text{Alt}_{2n}(321)} 1
\]

\[
\sum_{\text{Alt}_{4n+1}(321)} (-1)^{\text{inv}(\pi)} = (-1)^{n} \sum_{\text{Alt}_{2n}(321)} 1
\]

\[
\sum_{\text{Alt}_{4n}(321)} (-1)^{\text{inv}(\pi)} = 0
\]

\[
\sum_{\text{Alt}_{4n-1}(321)} (-1)^{\text{inv}(\pi)} = 0
\]
ARM on $\text{Alt}_n(321)$

- It is so similar to Simion-Schmidt’s result.
  - Hence, it is natural to seek the ARM identities.

**Theorem** [—, 2012]

- $\sum_{\text{Alt}_{4n+2}(321)} (-1)^{\text{inv}(\pi)} q^{\text{lead}(\pi)} = (-1)^{n+1} \sum_{\text{Alt}_{2n}(321)} q^{2 \cdot \text{lead}(\pi)}$
- $\sum_{\text{Alt}_{4n+1}(321)} (-1)^{\text{inv}(\pi)} q^{\text{lead}(\pi)} = (-1)^{n} \sum_{\text{Alt}_{2n}(321)} q^{2 \cdot \text{lead}(\pi)}$
- $\sum_{\text{Alt}_{4n}(321)} (-1)^{\text{inv}(\pi)} q^{\text{lead}(\pi)} = (-1)^{n+1}(1 - q) \sum_{\text{Alt}_{2n}(321)} q^{2(\text{lead}(\pi)-1)}$
- $\sum_{\text{Alt}_{4n-1}(321)} (-1)^{\text{inv}(\pi)} q^{\text{lead}(\pi)} = (-1)^{n}(1 - q) \sum_{\text{Alt}_{2n}(321)} q^{2(\text{lead}(\pi)-1)}$
Another

- ‘Ending’ \((\text{end}(\pi) := \pi_n)\) also works!

- **Theorem** [—, 2012]

\[
\sum_{\text{Alt}_{4n+2}(321)} (-1)^{\text{inv}(\pi)} q^{\text{end}(\pi)} = (-1)^{n+1} \sum_{\text{Alt}_{2n}(321)} q^{2 \cdot \text{end}(\pi)}
\]

\[
\sum_{\text{Alt}_{4n+1}(321)} (-1)^{\text{inv}(\pi)} q^{\text{end}(\pi)} = (-1)^{n} \sum_{\text{Alt}_{2n}(321)} q^{2 \cdot \text{end}(\pi)}
\]

\[
\sum_{\text{Alt}_{4n}(321)} (-1)^{\text{inv}(\pi)} q^{\text{end}(\pi)} = (-1)^{n} (1 - q) \sum_{\text{Alt}_{2n}(321)} q^{2(\text{end}(\pi) - 1)}
\]

\[
\sum_{\text{Alt}_{4n-1}(321)} (-1)^{\text{inv}(\pi)} q^{\text{end}(\pi)} = (-1)^{n+1} (1 - q) \sum_{\text{Alt}_{2n}(321)} q^{2(\text{end}(\pi) - 1)}
\]
Part 3

Baxter permutations (321)
Baxter permutations

• \( \pi \in S_n \) is Baxter if for all \( 1 \leq a < b < c < d \leq n \),
  • if \( \pi_a + 1 = \pi_d \) and \( \pi_b > \pi_d \), then \( \pi_c > \pi_d \).
  • if \( \pi_d + 1 = \pi_a \) and \( \pi_c > \pi_a \), then \( \pi_b > \pi_a \).

• For every 2 black dots whose height differ by 1,
  • ...the broken line can be found,
  • ...such that dots in between are in shaded area.
• 5321746 is **not** Baxter.

\[
\text{Theorem} \ [\text{Chung, Graham, Hoggatt, Kleitman}]
\]

\[
|\text{Bax}_n| = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}.
\]

• \(\text{Bax}_n := \) set of Baxter permutations of length \(n\).
Baxter permutation avoiding 123

· $\text{Bax}_n(123) := \text{Bax}_n$ and avoid 123.

**Theorem** [Mansour, Vajnovszki, 2007]

$$
\sum_{n \geq 0} |\text{Bax}_n(123)| z^n = \frac{1 - 2z + z^2}{1 - 3z + 2z^2 - z^3}.
$$

· $|\text{Bax}_n(123)|_{n \geq 0} = 1, 1, 2, 5, 12, 28, 65, \ldots$
· $|\text{Bax}_n(123)|_{n \geq 0} = p_{3n}$, a Padovan number, defined by
  
  $(p_0, p_1, p_2) = (1, 0, 0)$ and $p_n = p_{n-2} + p_{n-3}$.
· $|\text{Bax}_n(321)| = |\text{Bax}_n(123)|$
ARM on $\text{Bax}_n(321)$

- We have found an ARM-type identity on $\text{Bax}_n(321)$.

**Theorem** [—, 2012] For $n \geq 0$, we have

$$
\sum_{\pi \in \text{Bax}_{2n+1}(321)} (-1)^{\text{maj}(\pi)} p^{\text{fix}(\pi)} q^{\text{des}(\pi)} = p \cdot \sum_{\pi \in \text{Bax}_n(321)} p^{2 \cdot \text{fix}(\pi)} q^{2 \cdot \text{des}(\pi)}.
$$

- The sign is controlled by $\text{maj}$ (somewhat surprising)
For the even length, the corresponding ARM identity is a sum.

Theorem [—, 2012] For \( n \geq 0 \), we have

\[
\sum_{\pi \in \text{Bax}_{2n}(321)} (-1)^{\text{maj}(\pi)} p^{\text{fix}(\pi)} q^{\text{des}(\pi)} = (-1)^n \sum_{\pi \in \text{Bax}_n(321)} p^{2 \cdot \text{fix}(\pi)} q^{2 \cdot \text{des}(\pi)} - q \sum_{i=0}^{n-1} \sum_{\pi \in \text{Bax}_i(321)} p^{2 \cdot \text{fix}(\pi)} q^{2 \cdot \text{des}(\pi)}.\]


Part 4

Double Simsun permutations (312)
Simsun permutations

• $\pi \in \mathcal{S}_n$ is **simsun** if for all $1 \leq k \leq n$,
  - $\pi$ restricted to $\{1, 2, \ldots, k\}$ has no double descent.
  - 6274351 is **not** simsun,
    - 2431 has double descent 431.

• Let $|SS_n|$ be the simsun permutations of length $n$.
  - **Theorem** [Simion, Sundaram].
    $$|SS_n| = |\text{Alt}_{n+1}|.$$
Double simsun permutations

- $\pi$ is **double simsun** if both $\pi$ and $\pi^{-1}$ are simsun.
  - 51324 is simsun but **not** double simsun,
    - since $(51324)^{-1} = 24351$ is not simsun.
- Let $|DS_n| :=$ double simsun permutations of length $n$.
  - $|DS_n|$ is still unknown. However,....

**Theorem** [Chuang, Eu, Fu, Pan, Fundamenta Informaticae, to appear].

\[
\begin{align*}
|DS_n(123)| &= 2, \\
|DS_n(132)| &= S_n, \\
|DS_n(213)| &= S_n, \\
|DS_n(231)| &= 2^{n-1}, \\
|DS_n(312)| &= 2^{n-1}, \\
|DS_n(321)| &= \frac{1}{n+1} \binom{2n}{n},
\end{align*}
\]

- where $S_n$ is the ‘RNA secondary structure number’.
Theorem [—, 2012]. For $n \geq 1$, we have

$$\sum_{\pi \in \text{DS}_{2n+2}(312)} (-1)^{\text{maj}(\pi)} \cdot q^{\text{fix}(\pi)} = (-1 + q^2) \sum_{\pi \in \text{DS}_n(312)} q^{2\text{fix}(\pi)}.$$

Theorem [—, 2012]. For $n \geq 2$, we have

$$\sum_{\pi \in \text{DS}_{2n-1}(312)} (-1)^{\text{maj}(\pi)} \cdot q^{\text{lead}(\pi)} = \frac{2}{q(1 + q^2)} \sum_{\pi \in \text{DS}_n(312)} q^{2\text{lead}(\pi)}.$$
Part 5

Idea of Proof
Part 5-1

Sketch Proof: $\text{Alt}_n(321)$
**Alt}_{4n+2}(321) Sketch Proof**

- **Step 1:** Map everything to trees. $\text{Alt}_{2n} \longleftrightarrow T_n$
  - $T_n := \text{plane trees with } n \text{ edges.}$
  - $hsum(T) := \text{sum of heights of all nodes of } T.$
  - $\text{imp}(T) := \text{nodes on the leftmost path of } T$

```
31627485
31, 32, 62, 64
65, 74, 75, 85

inv(\pi) \longleftrightarrow hsum(T)
lead(\pi) \longleftrightarrow \text{imp}(T)
```
\textbf{Alt}_{4n+2}(321) Sketch Proof}

- **Step 2**: Look at identities on trees.

We are to prove

\[
\sum_{\text{Alt}_{10}(321)} (-1)^{\text{inv}(\pi)} q^{\text{lead}(\pi)} = (-1)^3 \sum_{\text{Alt}_4(321)} q^{2 \cdot \text{lead}(\pi)}
\]

which becomes

\[
\sum_{T_5} (-1)^{\text{hsum}(T)} q^{\text{lmp}(T)} = (-1) \sum_{T_2} q^{2 \cdot \text{lmp}(T)}
\]
**Step 3**: Devise an involution $\Phi$ (and a bijection $\Lambda$).

\[
\sum_{T_5} (-1)^{\text{hsum}(T)} q^{\text{imp}(T)} = (-1) \sum_{T_2} q^{2 \cdot \text{imp}(T)}
\]
$\text{Alt}_{4n+2}(321)$ Sketch Proof

- $\Phi$: an involution (to cancel w.r.t. hsum).
  - Find the last illegal vertex $v$ via postorder
    - illegal vertex: (i) “leaf, but is not the first child” or
      (ii) “inner vertex, but is the first child”.

- Do the following ($\text{hsum}$ will change sign, $\text{Imp}$ keep unchanged).
\textbf{Alt}_{4n+2}(321) Sketch Proof

\begin{itemize}
  \item If \( \not\exists \) illegal point, change the last (tree, vertex) \( \longleftrightarrow \) (vertex, tree).
  \item If \( \not\exists \) illegal point, \( \not\exists \) (tree, vertex), \( \not\exists \) (vertex, tree), \( \Rightarrow \) a fix point.
\end{itemize}
### $\text{Alt}_{4n+2}(321)$ Sketch Proof

<table>
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<th>$T_{5,2}$</th>
<th>$T_{5,3}$</th>
<th>$T_{5,4}$</th>
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</tr>
</thead>
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<td>$+$</td>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
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<td><img src="image4.png" alt="Diagram" /></td>
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<tr>
<td>$-$</td>
<td><img src="image6.png" alt="Diagram" /></td>
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<td><img src="image8.png" alt="Diagram" /></td>
<td><img src="image9.png" alt="Diagram" /></td>
<td><img src="image10.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

- The remaining task is to map fix points to $T_2$.
  - We need a bijection $\Lambda: T_2 \to T_5$
$\text{Alt}_{4n+2}(321)$ Sketch Proof

- $\Lambda$: a bijection (from $T_n$ to $T_{2n+1}$).

- Idea:
  - Double the points on the left most path.
  - For each subtree, append a single edge on each vertex.
\[ \sum_{T_5} (-1)^{hsum(T)} q^{lmp(T)} = (-1) \sum_{T_2} q^{2 \cdot lmp(T)} \]
**Alt$_{4n}(321)$ Sketch Proof**

- Proof of Alt$_{4n}(321)$:

\[
\sum_{T_4} (-1)^{hsum(T)} q^{\text{Imp}(T)} = - \sum_{T_2} q^{2(\text{Imp}(T) - 1)} + q \sum_{T_2} q^{2(\text{Imp}(T) - 1)}
\]
Part 5-2

Very sketched Proof: $\text{Bax}_n(321)$
Sketch proof of $\text{Bax}_n(321)$

- Done by multivariable generating functions.
  - Let $b_n := b_n(t, p, q) = \sum_{\pi \in \text{Bax}_n(321)} t^{\text{maj}(\pi)} p^{\text{fix}(\pi)} q^{\text{des}(\pi)}$.
  - We are to prove $b_{2n+1}(-1, p, q) = p \cdot b_n(1, p^2, q^2)$.

- Very sketched proof:
  - $g_n := b_n(1, p, q)$
    \[ g_n = (2 + p)g_{n-1} - (1 + 2p - q)g_{n-2} + p \cdot g_{n-3}. \]
  - $h_n := b_n(-1, p, q)$
    \[ h_{2n+1} = (2 + p^2)h_{2n-1} - (1 + 2p^2 - q^2)g_{2n-3} + p^2 \cdot h_{2n-5}. \]
  - $h_{1/3/5}(p, q) = p \cdot g_{0/1/2}(p^2, q^2)$ QED.

- **Problem:** Is there a combinatorial proof?
Part 5-3

Too sketched Proof: $DS_n(312)$
Sketch proof of $\text{DS}_n(312)$

• Done by an involution on compositions of $[n]$.

• Composition of $[n]$ with $\text{DS}_n(312)$: [Eu et al, 2012]
  $\cdot$ $3 + 4 + 2 \leftrightarrow 123|4567|89 \leftrightarrow 312745698$

• We are to prove
  $\sum_{\pi \in \text{DS}_{2n+2}(312)} (-1)^{\text{maj}(\pi)} \cdot q^{\text{fix}(\pi)} = (-1 + q^2) \sum_{\pi \in \text{DS}_n(312)} q^{2\text{fix}(\pi)}.$

  $\sum_{\pi \in \text{DS}_{2n-1}(312)} (-1)^{\text{maj}(\pi)} \cdot q^{\text{lead}(\pi)} = \frac{2}{q(1 + q^2)} \sum_{\pi \in \text{DS}_n(312)} q^{2\text{lead}(\pi)}.$

• Too sketched proof:
  $\cdot$ Observer what $\text{maj}$, $\text{fix}$, $\text{lead}$ means on compositions.
  $\cdot$ Design involutions (...not trivial...).

QED.
Part 6

Discussions
Naive believes were refuted over & over

- ...ARM appears only in $\mathfrak{S}_n(321)$?
  $\rightarrow$ NO, we have $\text{Alt}_n(321)$.

- ...ARM appears only in Catalan family?
  ...sign only controlled by inversion?
  $\rightarrow$ NO, we have $\text{Bax}_n(321)$.

- ...ARM appears only in 321-avoiding?
  $\rightarrow$ NO, we have $\text{DD}_n(312)$.

- ...If ARM appears, then $f(q)$ is a polynomial?
  $\rightarrow$ NO, $f(q) = \frac{2}{q(1+q^2)}$ in $\text{DD}_n(312)$.
General Setting

- Conceptually, we are searching for
  \[ C_n, (\text{stat}_2, \text{stat}_2) \]
  such that essentially the following holds:

  \[
  \sum_{C_{2n}} (-1)^{\text{stat}_1} q^{\text{stat}_2} = f(q) \sum_{C_n} q^{2 \cdot \text{stat}_2},
  \]

  where \( f(q) \) is some rational polynomial.

- In this talk we present
  - \( \text{Alt}_n(321), (\text{inv}, \text{lead}) \)
  - \( \text{Bax}_n(321), (\text{maj}, \text{des}) \)
  - \( \text{DS}_n(312), (\text{maj}, \text{fix}) \)
  - \( \text{DS}_n(312), (\text{maj}, \text{lead}) \)
Are there others?

- Yes, we have about another 30 nontrivial results.

- E.g.
  - \( \text{Alt}_n(321), (\text{chg}, \text{ldes}) \)
  - \( \text{Alt}_n(321), (\text{cochg}, \text{exc}) \)
  - \( \text{Bax}_n(321), (\text{chg}, \text{des}) \)
  - \( \text{Bax}_n(321), (\text{imaj}, \text{fix}) \)
  - \( \text{DS}_n(231), (\text{fix}, \text{end}) \)
  - \( \text{DS}_n(231), (\text{maj}, \text{inv}) \)
  - ...etc.

- However, ARM identities are relatively rare.
  - \( \text{C}_n, (\text{stat}_2, \text{stat}_2) \) **seldom** produces ARM.
Discussions

- **Encouraging:**
  We have a bunch of nontrivial identities.

- **Disappointing:**
  Proofs are done case by case.

- **Question:** Why is there ARM type identities?
  - What is lurking behind?
  - Is there a unifying/systematic approach?
  - Can it be generalized to, say, Coxter groups?

- We are working on these questions, with some progress.
Thanks

Flow gently, sweet Afton, amang thy green braes,
Flow gently, I’ll sing thee a song in thy praise.
— Robert Burns

Welcome any discussions and collaboration!