Small permutation classes

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Much of the early work in permutation patterns was motivated by the Stanley-Wilf Conjecture, which stated that every nontrivial permutation class has a finite *(upper) growth rate*,

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|},$$

where $\mathcal{C}_n$ denotes the set of permutations of length n in the permutation class $\mathcal{C}$. While Marcus and Tardos elegantly resolved this conjecture (in the affirmative) in 2004, we still know very little about these numbers. In particular, which numbers can occur as growth rates of permutation classes?

At Permutation Patterns 2007, I presented the following result, extending earlier work of Kaiser and Klazar [**?**].

Theorem 1 (Vatter [**?**]). *Let κ denote the unique positive root of $x^3 - 2x^2 - 1$, approximately* 2.20557*. If the upper growth rate of $\mathcal{C}$ is less than κ then $\mathcal{C}$ has a proper growth rate which is either* 0, 2*, a root of one of the four polynomials*

(P1) $x^3 - x^2 - x - 3$,

(P2) $x^4 - x^3 - x^2 - 2x - 3$,

(P3) $x^4 - x^3 - x^2 - 3x - 1$,

(P4) $x^5 - x^4 - x^3 - 2x^2 - 3x - 1$,

or a root of one of the three families of polynomials

(F1) $x^{k+1} - 2x^k + 1$,

(F2) $(x^3 - 2x^2 - 1)x^{k+\ell} + x^\ell + 1$, *or*

(F3) $(x^3 - 2x^2 - 1)x^k + 1$

for integers $k \geq 1$ and $\ell \geq 0$.

The number κ is the threshold of a sharp phase transition: there are only countably many permutation classes of growth rate less than κ, but uncountably many of growth rate κ. Furthermore, it is the first growth rate at which permutation classes may contain infinite antichains, which in turn is the cause of much more complicated structure. For this reason we single out classes of growth rate less than κ as *small.*

While Theorem **??** characterizes the asymptotics of small permutation classes, it does not give their fine structure, and in particular it says nothing about their exact enumeration. In this talk I will discuss recent joint work with Michael Albert and Nik Ruškuc, in which we were able to complete the structural characterization of small classes, leading to the following result.

Theorem 2 (Albert, Ruškuc, and Vatter [**?**]). *All small permutation classes have rational generating functions.*

The techniques involved in the proof of Theorem **??** involve the substitution decomposition, originally studied by Albert and Atkinson [**?**], and the geometric grid classes of Albert, Atkinson, Bouvel, Ruškuc, and Vatter [**?**].

[1] Michael H. Albert and M. D. Atkinson, *Simple permutations and pattern restricted permutations*, Discrete Math. **300** (2005), no. 1-3, 1–15.

[2] Michael H. Albert, M. D. Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter, *Geometric grid classes of permutations*, *Trans. Amer. Math. Soc.*, to appear.

[3] Michael H. Albert, Nik Ruškuc, and Vincent Vatter, *Inflations of geometric grid classes of permutations*, arXiv:1202.1833v1 [math.CO].

[4] Tomáš Kaiser and Martin Klazar, *On growth rates of closed permutation classes*, Electron. J. Combin. **9** (2003), no. 2, Research paper 10, 20 pp.

[5] Vincent Vatter, *Small permutation classes*, Proc. Lond. Math. Soc. (3) **103** (2011), 879–921.