Permuted Basement Fillings, k-ary Trees, and Watermelons

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Symmetric functions are important objects of study which illustrate the connection between algebra, representation theory, and combinatorics. In particular, the Schur functions are a notable basis for the symmetric functions because they have a combinatorial interpretation as the generating function of column-strict tableaux, as well as representation-theoretic value as the irreducible characters of the symmetric group. A q-t-analogue of the Schur functions are the symmetric Macdonald polynomials, introduced by Macdonald in 1988, from which the Schur functions can be obtained by setting q=t=0. Even more general are the nonsymmetric Macdonald polynomials, of which the symmetric Macdonald polynomials are a special case. In 2007, Haglund, Haiman, and Loehr gave a combinatorial interpretation to these nonsymmetric Macdonald polynomials, namely fillings of certain diagrams with positive integer entries. Since then, Mason has studied the polynomials $\widehat{E}_\gamma$ that result from setting q=t=0 in these nonsymmetric Macdonald polynomials. These can be considered a nonsymmetric refinement of the Schur functions, and are generated by fillings of certain diagrams, indexed by weak compositions, with basement permutation equal to the identity. When this basement is permuted to equal σ, we obtain the combinatorial objects of interest, known as *permuted basement fillings.* These fillings generate the polynomials $\widehat{E}_\gamma^\sigma$, which decompose the Schur functions.

Since the Schur functions are known to possess many nice properties, there is much interest in which elements of that structure are maintained by the the $\widehat{E}_\gamma^\sigma$s. Much of the study of PBFs has focused on cataloging the algebraic properties common to the Schur functions and $\widehat{E}_\gamma^\sigma$s, but not much work has been done on enumerating the permuted basement fillings, or PBFs, which generate the $\widehat{E}_\gamma^\sigma$s. One prominent question is whether there is an analogue of the hook formula for PBFs. Unfortunately, since the requirements for being a PBF are much stricter than the requirements for being a tableau, and some of these requirements are quite complicated, there does not seem to be anything analogous to the hook formula in this general situation. When we fix a simple enough shape, however, we can count the number of PBFs of that shape.

In this talk, I will discuss how one can count the number of PBFs of certain basic shapes, including all rectangular shapes. We will find that these objects, which have come to be a topic of study primarily because of their algebraic significance, also have connections to familiar combinatorial objects including k-ary trees, lattice paths, and watermelons. Aside from enumerating these permuted basement fillings, we will begin to look at certain statistics to find q-analogues of these results. For example, for a certain class of PBFs counted by k-ary trees, we will give a bijection to lattice paths and see how a descent between entries in the top row of a PBF corresponds to a certain behavior in the path. Further study of patterns and statistics within these PBFs seems likely to yield interesting results, as there is much yet to be discovered about these objects.