Random permutations (and beyond)

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In this talk we view uniform random permutations as part of a continuum of random mapping models and we investigate the component structure of the random mappings in this continuum as the mappings become (in some sense) more like permutations. Specifically, let $[n] = \{1, 2, ..., n\}$, let $\mathcal{M}_n$ denote the set of all mappings $f : [n] \to [n]$, and let $S_n \subset \mathcal{M}_n$ denote the set of all permutations $\sigma : [n] \to [n]$. Any mapping $f \in \mathcal{M}_n$ can be represented by a directed graph $G(f)$ on vertices labelled $1, 2, ..., n$ where there is a directed edge $i \to j$ in $G(f)$ if and only if $f(i) = j$. So if $\sigma \in S_n$, then $G(\sigma)$ is the directed graph that represents the cycle structure of σ and every vertex in $G(f)$ has in-degree 1. More generally, if $f \in \mathcal{M}_n$, then the connected components of $G(f)$ consists of directed cycles with directed trees attached to the cycles and vertices can have in-degree greater than 1. If T is a random element of $\mathcal{M}_n$, then $G(T)$ is a random directed graph and we can investigate random variables that are determined by the structure of the digraph $G(T)$. One such random variable is $C_1(T)$, the size of the component in $G(T)$ which contains the vertex labelled 1 (i.e. the size of a 'typical' component). It is well-known that is σ_n is a random permutation on $[n]$, then $C_1(\sigma_n)$ is uniformly distributed on $[n]$. In this talk we consider the exact and asymptotic distributions of $C_1(T_{n,a})$ where, for $0 \leq a \leq n$, $T_{n,a}$ is a random element of $\mathcal{M}_n$ such that the vertices in the digraph $G(T_{n,a})$ have at least $n - a$ vertices with in-degree 1 and at most a vertices with in-degree 2 and such that $T_{n,0} = \sigma_n$. (We note that, in some sense, the smaller the value of a relative to n, the 'closer' the random mapping $T_{n,a}$ is to the random permutation σ_n). The results obtained in this talk are based on urn scheme arguments and use a calculus developed by the authors for random mappings with exchangeable in-degrees.

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