

# Surprising Symmetries in Objects Counted by the Catalan numbers

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Let  $S_{n,r}(q)$  be the total number of copies of the pattern  $q$  in all  $r$ -avoiding permutations of length  $n$ . In this paper, we first prove the identities

$$S_{n,132}(312) = S_{n,132}(231) = S_{n,132}(213).$$

The first equality is trivial, but the second one is not. In fact, the two statistics in the second equality are *not* equidistributed, but they have the same cumulative value. A proof using generating functions is relatively straightforward, but we will also present a combinatorial proof. This is the first 3-fold symmetry in Catalan-like objects we have encountered.

Then we significantly generalize our results by presenting a large class of non-trivial equivalences in the above sense for patterns of arbitrary length. The proofs of these generalizations are combinatorial.