New Refinements of a Classical Formula in Consecutive Pattern Avoidance

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Let $\mathfrak{S}_n(12\cdots m)$ be the set of permutations of length n avoiding the monotone consecutive pattern $12\cdots m$. (Call these m-monotone avoiders.)

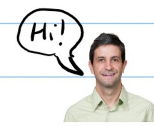
Thm. (David & Barton 1962) For $m \geq 2$,

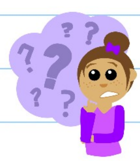
$$\sum_{n=0}^{\infty} |\mathfrak{S}_n(12\cdots m)| \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \left(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!} \right) \right]^{-1} .$$

$\downarrow$ refinement by inversion number!

Thm. (Elizalde 2016) Let $P^{inv}_{m,n}(q) = \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} q^{inv(\sigma)}$. For $m \geq 2$,

$$\sum_{n=0}^{\infty} P^{inv}_{m,n}(q) \frac{x^n}{[n]_q!} = \left[\sum_{n=0}^{\infty} \left(\frac{x^{mn}}{[mn]_q!} - \frac{x^{mn+1}}{[mn+1]_q!} \right) \right]^{-1} .$$





Big Question:
Can we refine by other permutation statistics??

Answer: Yes!! (At least for "inverses" of "shuffle-compatible" statistics.)

Given a permutation statistic st, let $ist(\sigma) = st(\sigma^{-1})$.

Thm. (Z. 2021+) Let $A_{m,n}(t,q) = \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} t^{ides(\sigma)+1} q^{imaj(\sigma)}$. For $m \geq 2$,

$$\sum_{n=0}^{\infty} \frac{A_{m,n}(t,q)}{(1-t)(1-qt)\cdots(1-q^n t)} x^n = 1 + \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left({k+jm-1 \brack k-1}_q x^{jm} - {k+jm \brack k-1}_q x^{jm+1} \right) \right]^{-1} t^k .$$

Outline of proof:

- We lift the Goulden-Jackson cluster method for permutations to the Malvenuto-Reutenauer algebra (FQSym).

- GJCM in FQSym applied to $12\cdots m$:

$$\underbrace{\sum_{n=0}^{\infty} \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} \sigma = \left[\sum_{n=0}^{\infty} (12\cdots(mn) - 12\cdots(mn+1)) \right]^{-1}} \qquad (*)$$

Can recover the David-Barton formula and Elizalde's q-analogue by applying standard homomorphisms

- If st is "shuffle-compatible", there is a homomorphism on FQSym for counting permutations by ist.

- Apply appropriate homomorphism to (*), do algebraic manipulations, and profit!

(Now for something a little more complicated...)

Given a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, we call $i \in \{2,3,\dots,n-1\}$ a peak if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$. Let $\mathrm{pk}(\sigma)$ be the number of peaks of σ.

Thm. (Z. 2021+) Let $P^{\mathrm{ipk}}_{m,n}(t) = \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} t^{\mathrm{ipk}(\sigma)+1}$. For $m \geq 2$,

$$\frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1+t}{1-t}\right)^{n+1} P^{\mathrm{ipk}}_{m,n}\left(\frac{4t}{(1+t)^2}\right) x^n = 1 + \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(c_{m,j,k} x^{jm} - c'_{m,j,k} x^{jm+1} \right) \right]^{-1} t^k$$

where $c_{m,j,k} = \begin{cases} 2 \sum_{i=1}^{k} \binom{i+jm-1}{i-1}\binom{jm-1}{k-i}, & \text{if } j \geq 1 \\ 1, & \text{if } j = 0 \end{cases}$

and $c'_{m,j,k} = 2 \sum_{i=1}^{k} \binom{i+jm}{i-1}\binom{jm}{k-i}$.

(We also have a formula for the "inverse left peak number" ilpk; omitted here.)

Conjecture: The polynomials $A_{m,n}(1,t) = \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} t^{\mathrm{ides}(\sigma)+1}$,

$P^{\mathrm{ipk}}_{m,n}(t) = \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} t^{\mathrm{ipk}(\sigma)+1}$, and $P^{\mathrm{ilpk}}_{m,n}(t) = \sum_{\sigma \in \mathfrak{S}_n(12\cdots m)} t^{\mathrm{ilpk}(\sigma)}$

have real roots only.

THANK YOU!