On SIF permutations avoiding a pattern

Michael D. Weiner

Penn State Altoona
mdw8@psu.edu

Permutation Patterns 2021

Joint work with Juan Gil

SIF permutations

A permutation on $[n]$ is SIF (stabilized-interval-free) if it does not stabilize any proper subinterval of $[n]$. They are enumerated by the sequence $1, 1, 2, 7, 34, 206, 1476, 12123, \ldots$ (A075834):[1]

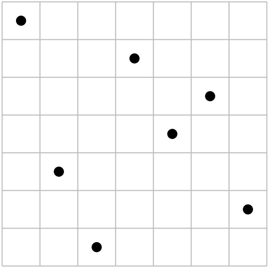
$$\begin{aligned} n = 1 &: \quad 1 \\ n = 2 &: \quad 21 \\ n = 3 &: \quad 231,\ 312 \\ n = 4 &: \quad 2341,\ 2413,\ 3142,\ 3412,\ 3421,\ 4123,\ 4312 \end{aligned}$$

By definition, any cycle is SIF, and for $n \geq 2$ every SIF permutation is fixed-point free.

[1] D. Callan, Counting stabilized-interval-free permutations, *J. Integer Seq.* **7** (2004)

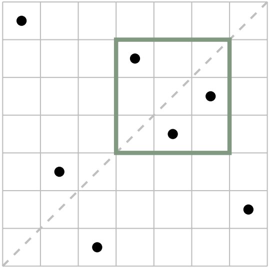
Example: 7316452

Not SIF!

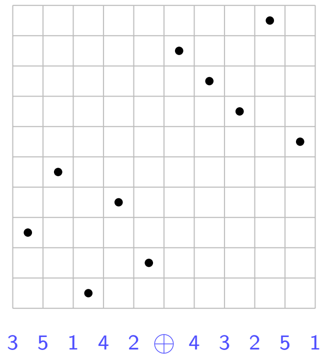


Example: 731<u>645</u>2

Not SIF!



Example: 3 5 1 4 2 9 8 7 10 6 (sum decomposable)



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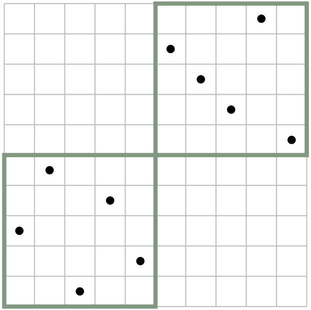

3 5 1 4 2 $\oplus$ 4 3 2 5 1

We can create all permutations by using the sum-indecomposable permutations as the “building blocks” via sum.

Let $P(x)$ be the OGF counting all permutations (according to size).

Let $I(x)$ be the OGF counting all sum-indecomposable permutations. (via the invert transform).

$$P(x) = \frac{1}{1 - I(x)}.$$

If we restrict our building blocks to sum-indecomposable permutations avoiding the pattern σ, we can generate all the permutations that avoid σ
(provided that σ itself is sum-indecomposable).

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We can create all permutations by using the SIF permutations as the “building blocks” as Dyck path ascents via nesting.

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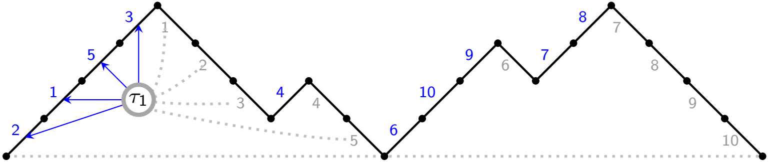
Let $S(x)$ be the OGF counting all SIF permutations (via the non-crossing partition transform).

$$P(x) = S(xP(x)).$$

Dyck path construction

Dyck path with ascents are colored by the SIF permutations $\tau_1 = 3412$, $\tau_2 = 1$, $\tau_3 = 231$, and $\tau_4 = 21$, respectively.



$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 5 & 1 & 4 & 2 & 9 & 8 & 7 & 10 & 6 \end{pmatrix}.$$

Two Questions:

Can we enumerate the SIF permutations that avoid a pattern σ?

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What happens when we color the Dyck path ascents with SIF permutations that avoid a pattern σ?

Easy picking:

Note: SIF property respects both the inverse action as well as the Reverse-Complement action.

Lemma

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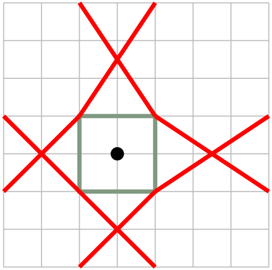

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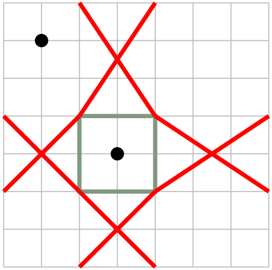

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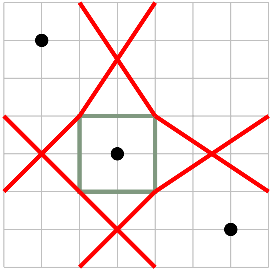

Avoiding a pattern of size 3

Pattern σ	$\lvert S_n^{\text{sif}}(\sigma)\rvert$
123	1, 1, 2, 5, 14, 44, 150, 496, 1758, 6018, 21782, 76414,...
132, 213, **321**	1, 1, 2, 5, 14, 42, 132, 429, 1430,...
231, 312	1, 1, 1, 2, 6, 18, 54, 170, 551, 1817, 6092, 20722, 71325,...

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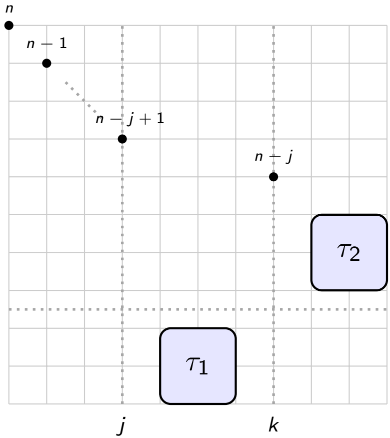
231-avoiding SIF permutations

An element of $B_{j,n,k}$ (not disjoint sets):



231-avoiding SIF permutations

For $n < 2j$, or $n = 2j$ and $k = j + 1$,

$$B_{j,n,k} = \{\sigma \in Av_n(231) | \sigma(r) = n{-}r{+}1 \text{ for } 1 \leq r \leq j, \sigma(k) = n{-}j\},$$

and for all other n, j, k,

$$B_{j,n,k} = \{\sigma \in Av_n^{\text{sif}}(231) | \sigma(r) = n{-}r{+}1 \text{ for } 1 \leq r \leq j, \sigma(k) = n{-}j\}.$$

Set $b_{j,n,k} = |B_{j,n,k}|$.

231-avoiding SIF permutations

1. $b_{n,2n,n+1} = C_{n-1}$
2. If $j < n < 2j$ and $j < k \leq n$ then $b_{j,n,k} = C_{n-k}C_{k-j-1}$
3. $b_{n,2n,2n} = C_{n-1}$
4. $b_{n,2n+1,2n+1} = C_n$
5. $b_{n-1,2n,n} = C_n - C_{n-1}$
6. For $n \notin \{2j-1, 2j\}$, $b_{j,n+1,n+1} = b_{j-1,n,j}$
7. For $n \in \{2j-1, 2j\}$, $b_{j,n+1,n+1} = b_{j-1,n,j} + C_{j-1}$
8. For $n > 2$, $b_{j,n+k-1,j+k} = b_{j,n,j+1}b_{j,j+k,j+k}$
9. For $n > j$, $\sum_{k=j}^{n-1} b_{j,n,k+1} = b_{j,n+1,n+1}$

231-avoiding SIF permutations

Theorem

With $\beta_j(x) = \sum_{n=j+1}^{\infty} b_{j,n,n} x^{n-j}$,

$$x - \left(1 + C_j x^{j+1}(x+1)\right)\beta_j(x) + \beta_j(x)\beta_{j+1}(x) = 0.$$

Also written as

$$\beta_j(x) = \frac{x}{1 + C_j x^{j+1}(x+1) - \beta_{j+1}(x)}.$$

Note: $\beta_1(x)$ is the generating function for the SIF avoiders of 231.

231-avoiding SIF permutations

Conjecture

The generating function is given by the continuing fraction

$$f(x) = \cfrac{x}{1 + C_1 x^2 (x+1) - \cfrac{x}{1 + C_2 x^3 (x+1) - \cfrac{x}{1 + C_3 x^4 (x+1) - \dots}}}$$

where C_n is the n-th Catalan number.

231-avoiding SIF permutations

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i.e.

$$\beta_j(x) = \cfrac{x}{1 + C_j x^{j+1}(x+1) - \cfrac{x}{1+C_{j+1}x^{j+2}(x+1) - \cfrac{x}{1+C_{j+2}x^{j+3}(x+1)-\dots}}}.$$

Avoiding a pair of patterns of size 3

(σ_1, σ_2)	$\lvert S_n^{\mathrm{sif}}(\sigma_1, \sigma_2)\rvert$	OEIS
123, 132	1, 1, 2, 3, 6, 9, 18, 27, ...	A182522
123, 231	1, 1, 1, 1, 2, 3, 3, 5, ...	N/A
123, 321	1, 1, 2, 3, 0, 0, 0, 0, ...	finite
132, 213	1, 1, 2, 5, 8, 17, 26, 53, ...	A62318
132, 231	1, 1, 1, 2, 4, 8, 16, 32, ...	A011782
132, 321	1, 1, 2, 3, 4, 5, 6, 7, ...	A028310
231, 312	1, 1, 0, 0, 0, 0, 0, 0, ...	finite
231, 321	1, 1, 1, 1, 1, 1, 1, 1, ...	A000012

(123,231)-avoiding SIF permutations

Theorem

For $n > 2$, the number of such permutations is given by

$$\begin{cases} \frac{(n+12)(n-2)}{24} & \text{if } n \equiv 0 \mod 6 \\ \frac{(n+5)(n-1)}{24} & \text{if } n \equiv 1 \mod 6 \\ \frac{(n+12)(n-2)}{24} & \text{if } n \equiv 2 \mod 6 \\ \frac{(n+3)(n+1)}{24} & \text{if } n \equiv 3 \mod 6 \\ \frac{(n+12)(n-2)}{24} - \frac{1}{3} & \text{if } n \equiv 4 \mod 6 \\ \frac{(n+3)(n+1)}{24} & \text{if } n \equiv 5 \mod 6 \end{cases}.$$

Generating function: $f(x) = 1 + x + x^2 - \frac{1-2x^2-2x^3+2x^5}{(1-x)^3(1+x)^2(1+x+x^2)}$.

References

Thank you!

- D. Callan, Counting stabilized-interval-free permutations, *J. Integer Seq.* **7** (2004), Article 04.1.8.
- N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.