On pattern avoidance in matchings and involutions

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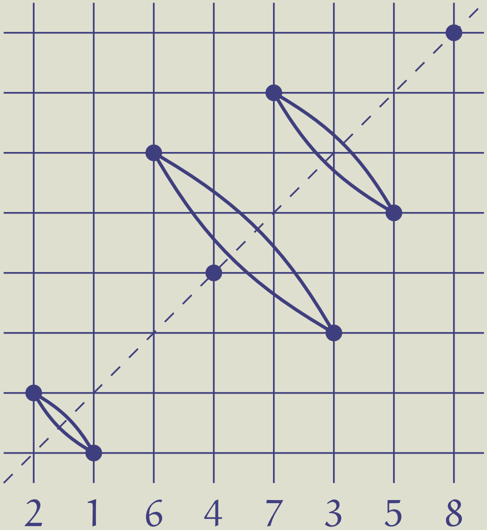
An involution is a permutation whose cycles are all 1-cycles and 2-cycles; equivalently, its diagram is fixed by reflection across the main diagonal.



$=$

$= (12)(36)(4)(57)(8).$

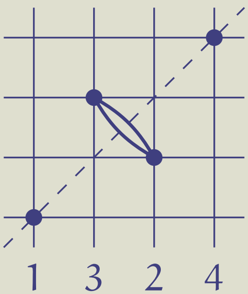
For involutions ρ and τ, we say τ $\mathfrak{I}$-contains ρ if ρ can be obtained from τ by some combination of these steps:

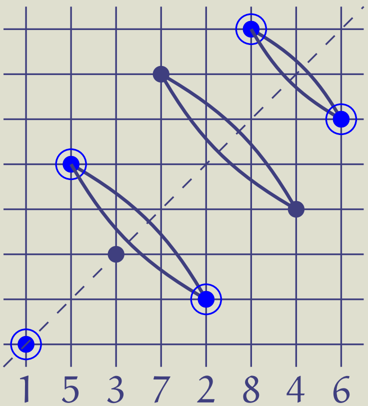
(1) Deleting a 2-cycle and standardizing;

(2) Deleting a 1-cycle (fixed point) and standardizing;

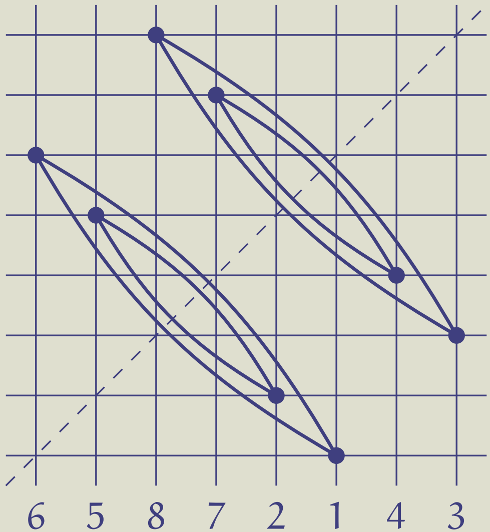
(3) Deleting one entry from a 2-cycle $(i, i+1)$ and standardizing.

$15372846 = (1)(25)(3)(47)(68)$ $\mathfrak{I}$-contains $1324 = (1)(23)(4)$:





- $\mathfrak{S}(\pi)$ is the set of permutations avoiding π.
- $\mathfrak{S}(\pi) \cap \mathfrak{I}$ is the set of involutions avoiding π in the sense of permutation patterns (this is well-studied).
- Define $\mathfrak{I}(\tau)$ to be the set of involutions that $\mathfrak{I}$-avoid the involution τ.



$65872143 \notin \mathfrak{S}(2143) \cap \mathfrak{I}$, but $65872143 \in \mathfrak{I}(2143)$.

- $\mathfrak{S}(\tau) \cap \mathfrak{I} \subseteq \mathfrak{I}(\tau)$: if an involution $\mathfrak{I}$-contains τ, then it contains τ as a permutation.
- $\mathfrak{S}(\tau) \cap \mathfrak{I} = \mathfrak{I}(\tau)$ holds for some τ but not others.
  - $\mathfrak{S}(4321) \cap \mathfrak{I} = \mathfrak{I}(4321)$, and $\mathfrak{S}(3412) \cap \mathfrak{I} = \mathfrak{I}(3412)$.
  - $\mathfrak{S}(2143) \cap \mathfrak{I} \subsetneq \mathfrak{I}(2143)$.

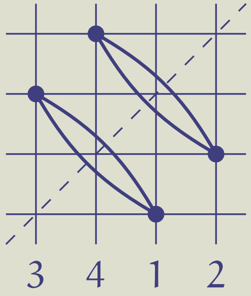
Theorem (Fang, Hamaker, & T. 2020): The following are equivalent:

(i) $\mathfrak{S}(\tau) \cap \mathfrak{I} = \mathfrak{I}(\tau)$;

(ii) $\tau \in \mathfrak{I}(12)$;

(iii) τ has the form $\tau = \sigma \ominus \sigma^{-1}$ or $\tau = \sigma \ominus 1 \ominus \sigma^{-1}$.

For instance, $3412 = 12 \ominus 12$, so $\mathfrak{S}(\tau) \cap \mathfrak{I} = \mathfrak{I}(\tau)$.



Corollary (Fang, Hamaker, & T. 2020): $|\mathfrak{I}_n(\tau)|$ is either bounded above by an exponential function or bounded below by $\lfloor n/2 \rfloor !$.

- This is because $|\mathfrak{I}_n(12)| = \lfloor n/2 \rfloor !$.

τ	$\lvert\mathfrak{I}_n(\tau)\rvert$
12	$= (n/2)!$
132	$\sim \frac{1}{2} e^{1/8}\, e^{\sqrt{n/2}}\, (n/2)!$
2143	$\sim \frac{1}{2} e^{-1/2}\, e^{\sqrt{2n}}\, (n/2)!$
123	$\sim \sqrt{\frac{2\pi}{27} n} \left(\frac{2}{\sqrt{3}}\right)^n (n/2)!$

(valid for even n)

Theorem (Fang, Hamaker, & T. 2020):

(a) $|\mathfrak{I}_n(321)| = |\mathfrak{S}_n(321) \cap \mathfrak{I}_n| = \binom{n}{\lfloor n/2 \rfloor}$.

(b) $|\mathfrak{I}_n(213)| = |\mathfrak{I}_n(132)| = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} k!$.

(c) $|\mathfrak{I}_n(123)| = \sum_{k=1}^{n} \lfloor \frac{k}{2} \rfloor! \, \lfloor \frac{n-k}{2} \rfloor! \binom{n - \lfloor \frac{k}{2} \rfloor - 1}{n-k}$.

Remark: We prove that an involution is in $\mathfrak{I}(123)$ if and only if it is the union of two involutions in $\mathfrak{I}(12)$. This is analogous to the classical characterization of $\mathfrak{S}(123)$ as the set of permutations that are a union of two permutations in $\mathfrak{S}(12)$, i.e. decreasing subsequences.

Future directions:

- If $\tau \in \mathfrak{I}(12)$, understand the relationship between the growth rates of $\mathfrak{I}(\tau)$ and $\mathfrak{S}(\tau)$.
- If $\tau \notin \mathfrak{I}(12)$, understand $\lim_{n\to\infty} \left(\frac{|\mathfrak{I}_n(\tau)|}{\lfloor n/2 \rfloor !}\right)^{1/n}$ (it must be between 1 and $\sqrt{2}$).
- Count $\mathfrak{I}(\tau)$ for $\tau \in \mathfrak{I}_4$.
- Count $\mathfrak{I}(1 \ldots m)$ for $m \geqslant 4$.