An Algorithm for Counting The Admissible Orderings of a Pinnacle Set

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Pinnacle sets and pinnacle orderings

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- **Ex:** If $S = \{3, 5, 7\}$ then $\sigma = 537$ is an admissible ordering as witnessed by $\pi = 4513276$. However, $\tau = 375$ is not admissible. Because pinnacles must have elements on either side, the pinnacles must appear in positions $2, 4$ and 6 in the given order. Since for 6 not to be a pinnacle, it must be directly to the left or right of 7. Since 6 is larger than both 3 and 5, either placement will force the other adjacent position not to be a pinnacle.

Counting admissible pinnacle orderings

- We can count admissible pinnacle orderings of an admissible pinnacle set S using the formula

$$\#\mathcal{O}(S) = \sum_{B \subseteq D' \colon\ |B| = d-1} \delta_{B \cup \{1_l, 1_r\}} \prod_{i=0}^{d-2} (d + 1 - i - r_{d-1-i}) \,.$$

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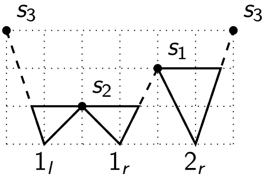
- The sum deals with (as in Quinn's talk) the concept of dales. A *dale* is the set of all elements between two pinnacles that are smaller than both pinnacles. We sum over every possible dale set, which corresponds to every cyclic ordering of the pinnacle set.



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- In addition to the formula above, we have found an improved formula that sums over compositions rather than the B subsets.

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Thank you!