The Generating Function of the Statistic des for Cyclic Permutations which avoid the patterns [1324] and [1423]

Jamie Schmidt

Joint work with: Rachel Domagalski, Jinting Liang, Quinn Minnich, Dr. Bruce Sagan, and Alexander Sietsema

Michigan State University

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Cyclic Permutations

Let's quickly review the necessary definitions.

- A *cyclic permutation* is a permutation where we let the end "wrap around" to the beginning. We denote this with brackets, like [123] instead of 123.
  - **Ex:** The only two cyclic permutations of length 3 are [123] and [132].
- We can consider pattern avoidance in this fashion.
  - **Ex:** In the linear version, $\pi = 21534$ avoided 321. If we consider $[\pi]$ as a cyclic permutation, it no longer does — $[\pi]$ contains [541] as a copy of [321].
- We usually standardize our cyclic permutations to begin with 1.
- We say that the set of permutations in the symmetric group S_n that avoid a particular set of patterns P is $\text{Av}_n(P)$.

Cyclic Statistics

- A *cyclic descent* is a descent in a cyclic permutation.
  - **Ex:** The cyclic permutation [1736425] has cyclic descents which start at the numbers 7, 6, 4, and 5.
- We denote the number of cyclic descents in a cyclic permutation $[\sigma]$ as $\mathrm{cdes}[\sigma]$.

The Theorem

Theorem

The generating function for the number of cyclic permutations on $[n] = \{1, 2, 3, \ldots, n\}$ *which avoid the patterns* [1324] *and* [1423] *is*

$$\sum_{[\sigma] \in Av_n([1324],[1423])} x^{cdes[\sigma]} = x(x+1)^{n-2}.$$

For simplicity's sake, we will say that $\mathrm{Av}_n = \mathrm{Av}_n([1324], [1423])$.

- We will proceed by induction.
- The base case for $n = 2$ is simple.
- But before we go on, we need to characterize the avoidance class.

The Theorem

Theorem

Every permutation in Av_{n-1} produces two permutations in Av_n through a method where n is inserted either before or after $n-1$, and no other permutations are possible.

- Because 3 and 4 are not adjacent in either pattern, inserting n next to $(n-1)$ for any cyclic permutation in Av_{n-1} cannot create either pattern.
- If inserting n did somehow create a pattern, we would also have a copy of that pattern with $(n-1)$ substituted for n.
- Moreover, we if we insert n at a position not next to $(n-1)$, then we create a cyclic permutation of the form $[\pi'] = 1 \dots (n) \dots x \dots (n-1)$ or $[\pi'] = 1 \dots (n-1) \dots x \dots (n)$ for at least one x.
- Then we have a copy of 1423 given by $1(n)x(n-1)$ or a copy of 1324 given by $1(n-1)x(n)$, respectively.

The Theorem

Theorem

The generating function for the number of cyclic permutations on $[n]$ *which avoid the patterns* $[1324]$ *and* $[1423]$ *is*

$$\sum_{[\sigma]\in Av_n} x^{cdes[\sigma]} = x(x+1)^{n-2}.$$

- Inserting n before $(n-1)$ will increase the number of descents in the cyclic permutation by one, as the numbers to the left and right of $(n-1)$ are both less than it, while inserting n after $(n-1)$ will keep the number of descents the same.
- The generating function follows, and the proof is complete.

Further Work in Cyclic Permutations

- We have also proven many other generating functions, and derived a single bivariate generating function for cdes and cpk. Check out our paper to see them!
- Other work includes results for counting avoidance sets for all doubles, triples, and quadruples of cyclic patterns of length 4, analysing cyclic shuffle-compatibility, and (coming soon!) counting pinnacle sets, their admissible orderings, and the permutations that have them.

Acknowledgements and References

You can find our cyclic pattern containment paper at https://arxiv.org/abs/2106.02534, or at this QR code:

Thanks for listening!