Hardness of $\mathcal{C}$-Permutation Pattern Matching

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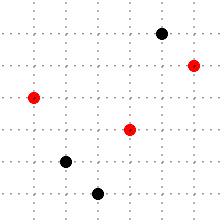
Pattern matching

PERMUTATION PATTERN MATCHING (PPM)
Input: Permutations π of size k and τ of size n.
Question: Is π contained in τ?



A pattern 213 is contained in a permutation 421365.

Theorem (Bose, Buss and Lubiw, 1998)

PERMUTATION PATTERN MATCHING *is NP-complete.*

Pattern matching restricted

$\mathcal{C}$-Permutation Pattern Matching ($\mathcal{C}$-PPM)
Input: Permutations $\pi \in \mathcal{C}$ of size k and $\pi \in \mathcal{C}$ of size n.
Question: Is π contained in τ?

Known cases

- $\mathrm{Av}(21)$-PPM – polynomial (trivial)
- $\mathrm{Av}(132)$-PPM – polynomial (Bose, Buss and Lubiw, 1998),
- $Av(321)$-PPM – polynomial (Guillemot and Vialette, 2009),
- $Av(3412, 2143)$-PPM – polynomial (Albert et al., 2016),
- $Av(4321)$-PPM – NP-complete (Jelínek and Kynčl, 2017), and
- $Av(\sigma)$-PPM with $|\sigma| \geq 10$ – NP-complete (Jelínek and Kynčl, 2017).

Grid classes

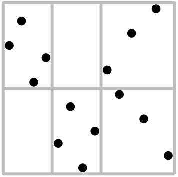
Definition

A $k \times \ell$ matrix $\mathcal{M}$ of permutation classes is a gridding matrix. The grid class of $\mathcal{M}$, denoted by Grid($\mathcal{M}$), is a class of permutations that can be partitioned, by $\ell - 1$ horizontal and $k - 1$ vertical cuts, into a $k \times \ell$ array of cells, where the cell in the i-th column and j-th row induces a pattern from $\mathcal{M}_{i,j}$.

Definition

The cell graph of $\mathcal{M}$ is a graph whose vertices are the cells of $\mathcal{M}$ that contain an infinite class, with two vertices being adjacent if they share a row or a column of $\mathcal{M}$ and all cells between them are empty.

$$\mathcal{M} = \begin{pmatrix} \mathrm{Av}(132) & & \mathrm{Av}(21) \\ & \mathrm{Av}(321) & \mathrm{Av}(12) \end{pmatrix}$$



Definition

A class $\mathcal{C}$ is monotone-griddable if $\mathcal{C} \subseteq$ Grid($\mathcal{M}$) for $\mathcal{M} \in \{\boxslash, \boxbslash, \emptyset\}^{k \times \ell}$.

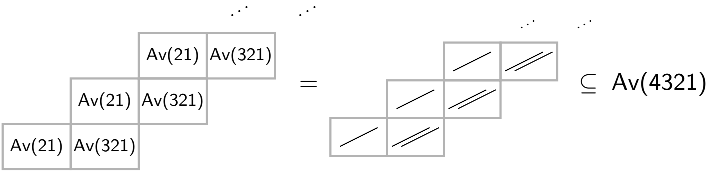
The hardness reduction

Definition

A class $\mathcal{C}$ has the $\mathcal{D}$-rich path property for a class $\mathcal{D}$, if there exists $\epsilon > 0$ such that for every k, $\mathcal{C}$ contains a grid subclass whose cell graph is a path of length k with no three cells in the same row or column and with at least $\epsilon \cdot k$ $\mathcal{D}$-entries.

Example

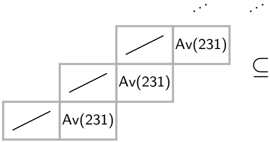
The class Av(4321) has the Av(321)-rich path property.



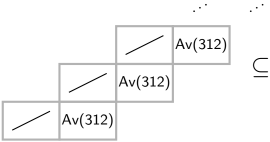
Theorem

Let $\mathcal{C}$ be a permutation class with the $\mathcal{D}$-rich path property for a non-monotone-griddable class $\mathcal{D}$. Then $\mathcal{C}$-PPM is NP-complete.

Applying the reduction



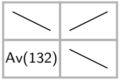
$\subseteq$ Av(4231),



$\subseteq$ Av(4312)

Corollary

If $\mathcal{M}$ is a gridding matrix such that $G_{\mathcal{M}}$ contains a cycle with no three vertices in the same row or column and one entry equal to a non-monotone-griddable class $\mathcal{D}$ then Grid($\mathcal{M}$)-PPM *is NP-complete*

Av(14523) $\supseteq$ 

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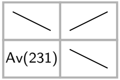
Av(24513) $\supseteq$ 

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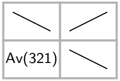
Av(32154, 42513) $\supseteq$ 

Main results

Theorem

If σ is a permutation of length at least 4 that is not symmetric to any of 3412, 3142, 4213, 4123 or 41352, then $\mathrm{Av}(\sigma)$-PPM is NP-complete.

Theorem

$\mathcal{C}$-PPM is polynomial-time solvable for any monotone-griddable class $\mathcal{C}$.

Question

What about $\mathrm{Av}(\sigma)$-PPM for $\sigma \in \{3412, 3142, 4213, 4123, 41352\}$?

Thank you!