Counting Permutations with a Given Pinnacle Set

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Pinnacle Sets

Definition

If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\text{Pin}\ \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

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Ex: $n = 9$

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In 2018, Davis et al. gave an explicit formula for pinnacle sets containing at most two pinnacles, but most of their methods for larger pinnacle sets were recursive. Then in 2021, Diaz-Lopez et al. found a non-recursive formula for any number of pinnacles, but it was still very slow for permutations in which the pinnacles were far apart.

Notation

Fix $n > 0$ and let S_n be the symmetric group on n elements. Suppose we have a pinnacle set $S = \{s_1 < s_2 < \ldots < s_d\}$ in S_n. We use the convention $s_0 = 0$ and $s_{d+1} = n + 1$ and for $0 \leq i \leq d$ let

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$$n_i = s_{i+1} - s_i - 1.$$

The n_i can be thought of as the number of non-pinnacle values between the pinnacles. For example, if $n = 6$ and $S = \{3, 5\}$, then $n_0 = 2, n_1 = 1$, and $n_2 = 1$

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One way to generate permutations having S as a pinnacle set is to fix an ordering of the pinnacles and then insert the non-pinnacles between them.

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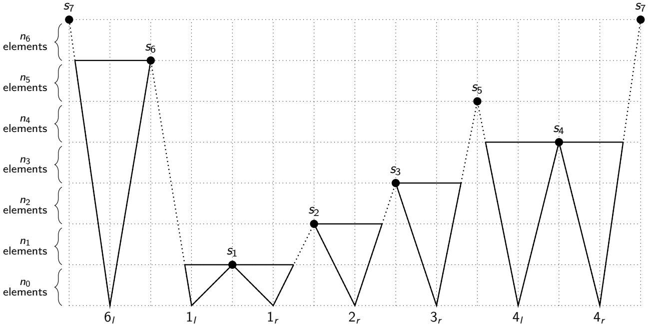

The diagram above shows a possible ordering of the pinnacle values and the diagonal lines between them represent where the non-pinnacles may be placed without becoming pinnacles themselves.

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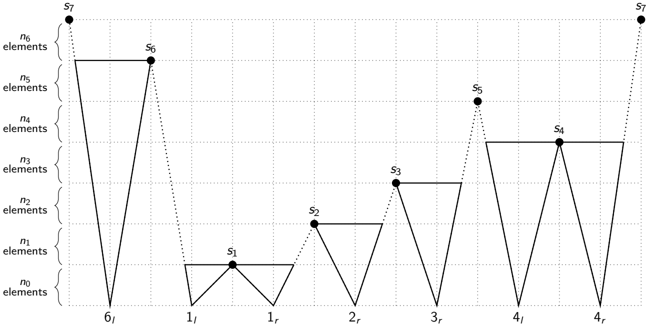

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Each solid triangle is called a *dale*. Each dale has a *rank* derived from the smaller pinnacle that bounds it, and also a "*l*" or "*r*" designation based on whether the dale is to the left or right of the smaller pinnacle.

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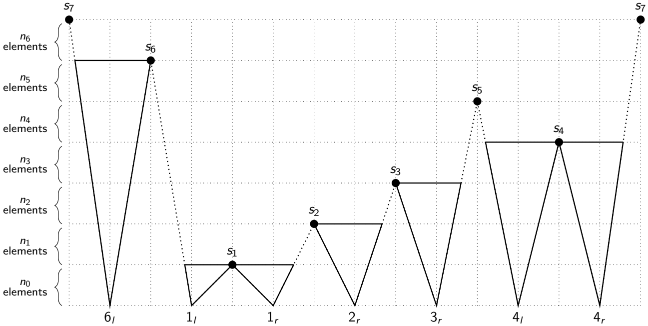

Each dale must contain at least one element in order to guarantee the s_i all become pinnacles. Therefore, we use the principle of inclusion and exclusion to ensure all dales end up non-empty, and to do this we need to be able to refer to arbitrary possible subsets of dales.

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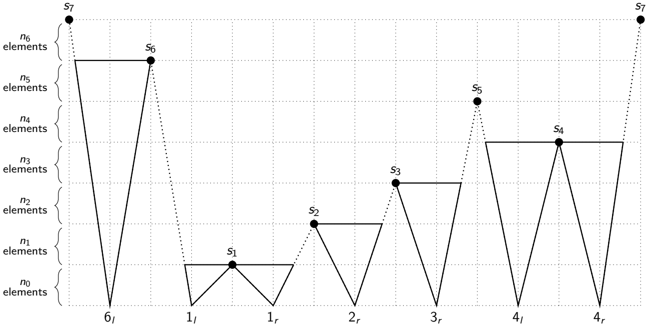

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Let $D = \{1_l, 1_r, 2_l, 2_r, \ldots, d_l, d_r\}$ be the set of all possible dales with partial order $1_l < 1_r < 2_l < \ldots < d_l < d_r$. For example, if there were 4 pinnacle values, then $D = \{1_l, 1_r, 2_l, 2_r, 3_l, 3_r, 4_l, 4_r\}$

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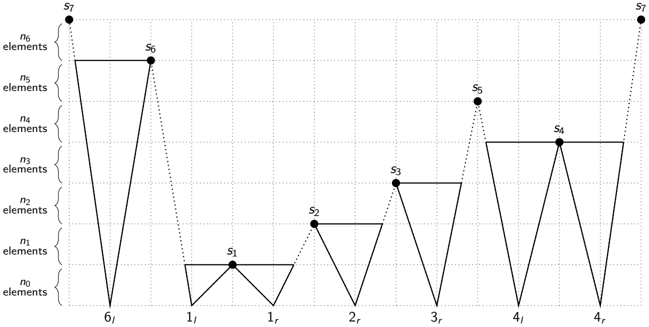

Given a $B \subset D$ with $|B| = b$, we define $r_j =$ the rank of the jth smallest element of B for $1 \leq j \leq b$.

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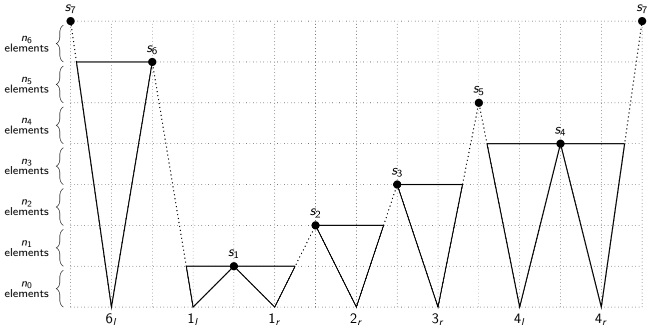

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For example, if $B = \{1_l, 3_l, 3_r, 4_r\} \subset D$, then $r_1 = 1, r_2 = 3, r_3 = 3, r_4 = 4$

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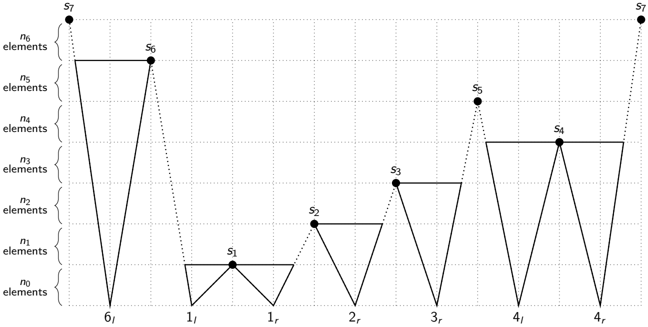

Additionally, given a $B \subset D$ with $|B| = b$, we define $b_i =$ the number of elements in B with rank at least i.

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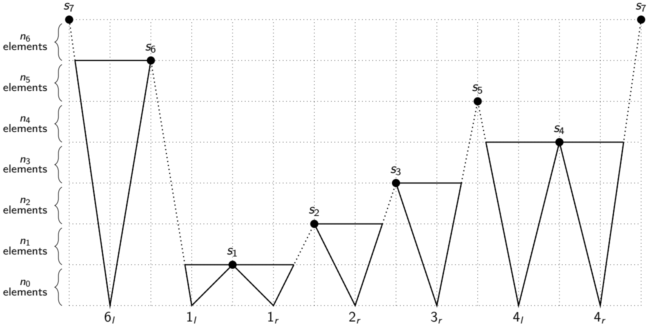

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For example, if $B = \{1_l, 3_l, 3_r, 4_r\} \subset D$, then $b_1 = 4, b_2 = 3, b_3 = 3, b_4 = 1$

A New Formula

Recall:

- $D = \{1_l, 1_r, 2_l, 2_r, \ldots, d_l, d_r\}$.
- $n_i = s_{i+1} - s_i - 1$.
- $r_j =$ the rank of the jth smallest element of $B \subset D$.
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Theorem

For $n > 0$, the number of permutations $\pi \in \mathfrak{S}_n$ with pinnacle set $S = \{s_1 < s_2 < \ldots < s_d\}$ is given by the following formula.

$$2^{n-2d-1} \sum_{B \subseteq D:\ |B| \leq d} (-1)^b (d-b)! \left(\prod_{i=0}^{b-1} (d+1-i-r_{b-i}) \right) \left(\prod_{i=0}^{d} (d+1-i-b_{i+1})^{n_i} \right).$$

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There is also an improved version of this formula that replaces the subsets B with certain compositions, ultimately resulting in fewer terms to sum over. Additionally, just this month, Falque, Novelli, and Thibon released a recursive algorithm which is a low degree polynomial in both n and d.

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References

Robert Davis, Sarah A. Nelson, T. Kyle Petersen, and Bridget E. Tenner. The pinnacle set of a permutation. *Discrete Mathematics*, 341(11):3249–3270, 2018.

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Justice Falque, Jean-Christophe Novelli, and Jean-Yves Thibon. Pinnacle Sets Revisited. arXiv:2106.05248v1.