Pattern Avoidance in Cyclic Permutations

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Cyclic Permutations

- A *cyclic permutation* is an equivalence class of linear permutations under rotation.

$$[\pi] = \{\pi_1\pi_2\ldots\pi_n,\ \ \pi_2\ldots\pi_n\pi_1,\ \ \ldots,\ \ \pi_n\pi_1\ldots\pi_{n-1}\}.$$

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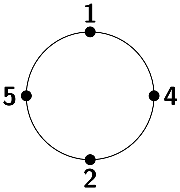
  - **Ex:** For example, $[1425] = \{1425,\ 4251,\ 2514,\ 5142\} = [5142]$.



Both [125] and [542] are subsequences.

Pattern Containment

- We say that $[\pi]$ *contains a copy of* $[\sigma]$ if $[\pi]$ has a subsequence of the same length and the same relative order as $[\sigma]$; $[\pi]$ *avoids* the pattern $[\sigma]$ if no subsequence of $[\pi]$ has the same relative order as $[\sigma]$.

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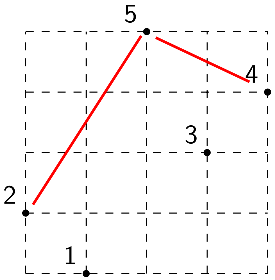
  - **Ex:** $[\pi] = [53421]$ contains $[\sigma] = [132]$ as a pattern because the subsequences $[254]$ (among others) has the same relative order as $[\sigma]$.



A copy of [132]

Pattern Avoidance

- There are only six cyclic permutations of length 4; Callan [1] showed that:

$$\#\mathrm{Av}_n([1234]) = \#\mathrm{Av}_n([1432]) = 2^n + 1 - 2n - \binom{n}{3};$$
$$\#\mathrm{Av}_n([1324]) = \#\mathrm{Av}_n([1423]) = F_{2n-3} \text{ (Fibonacci numbers)};$$
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- Our work is on counting the avoidance sets for sets of length 4 patterns.

Our Results (pairs)

- There are 15 pairs of cyclic patterns of length 4. From trivial Wilf equivalences, we may reduce to counting avoidance of the pairs:
  - $\#\text{Av}_n([1234],[1243]) = 2(n-2)$ for $n > 2$.
  - $\#\text{Av}_n([1234],[1324]) = 2(n-2)$ for $n > 2$.
  - $\#\text{Av}_n([1234],[1423]) = 1 + \binom{n-1}{2}$ for all n.
  - $\#\text{Av}_n([1243],[1324]) = 1 + \binom{n-1}{2}$ for all n.
  - $\#\text{Av}_n([1324],[1423]) = 2^{n-2}$ for $n > 1$.
  - $\#\text{Av}_n([1243],[1342]) = 4$ for $n > 3$.
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Theorem (A Cyclic Variant of the Erdős–Szekeres Theorem)

A cyclic sequence with length at least $rs + 2$ must contain an increasing subsequence of length $r + 2$ or a decreasing subsequence of length $s + 2$.

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- Other work includes discussing generating functions for cyclic permutation statistics.

References

David Callan.
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P.Erdős and G.Szekeres.
A combinatorial theorem in geometry.
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Preprint
https://arxiv.org/abs/2106.02534

Thank You!

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