Permutation groups and permutation patterns

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Proposition

For any $\ell, n \in \mathbb{N}_+$, if S is a subgroup of S_ℓ, then $\mathrm{Av}_n(S_\ell \setminus S)$ is a subgroup of S_n.

Notation: $\mathrm{Comp}^{(n)} S := \mathrm{Av}_n(S_\ell \setminus S)$.

The permutation groups of the form $\mathrm{Comp}^{(n)} S$ for some $S \subseteq S_\ell$, $\ell < n$, were characterized as the automorphism groups of certain relational structures.

E. LEHTONEN, R. PÖSCHEL,
Permutation groups, pattern involvement, and Galois connections,
Acta Sci. Math. (Szeged) **83** (2017) 355–375.

Group classes

A permutation class C in which every level C_n is a permutation group is called a **group class**.

M. D. ATKINSON, R. BEALS,
Permuting mechanisms and closed classes of permutations,
in: C. S. Calude, M. J. Dinneen (eds.), *Combinatorics, Computation & Logic,* Proc. DMTCS '99 and CATS '99 (Auckland), Aust. Comput. Sci. Commun., 21, No. 3, Springer, Singapore, 1999, pp. 117–127.

M. D. ATKINSON, R. BEALS,
Permutation involvement and groups,
Q. J. Math. **52** (2001) 415–421.

Group classes

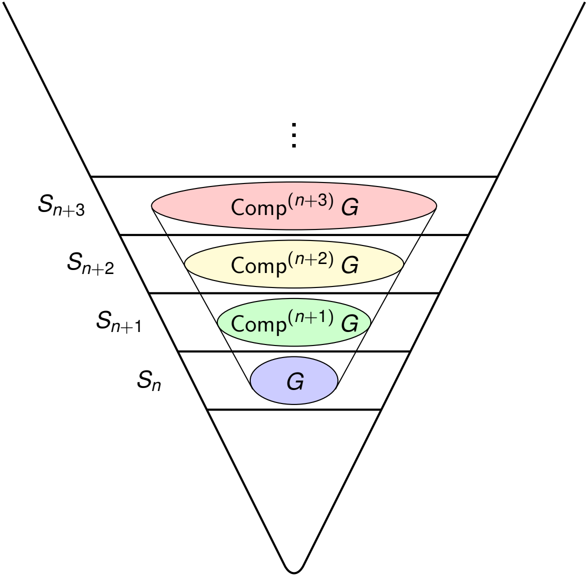
Theorem (Atkinson, Beals)

If C is a group class, then the level sequence $C_1, C_2, C_3, \ldots$ eventually coincides with one of the following ***stable families of groups:***

1. *the groups $S_n^{a,b}$ for some fixed $a, b \in \mathbb{N}_+$,*
2. *the natural cyclic groups Z_n,*
3. *the full symmetric groups S_n,*
4. *the groups $\langle G_n, \delta_n \rangle$, where $(G_n)_{n \in \mathbb{N}}$ is one of the above families (with $a = b$ in (1)).*

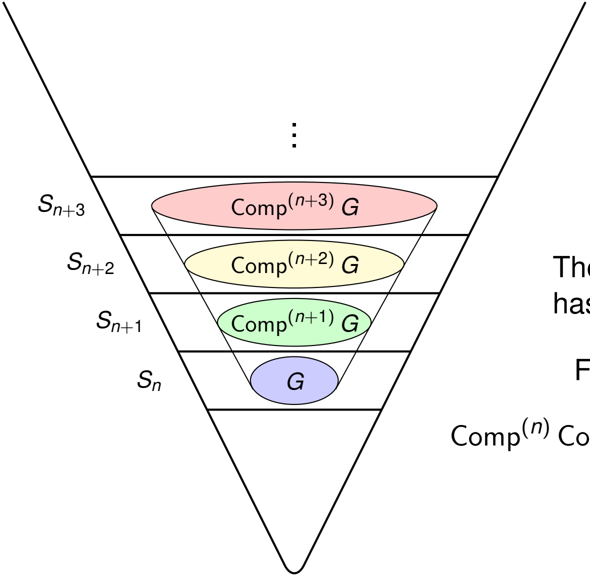
Atkinson and Beals also described explicitly the group classes in which every level is a transitive group.

Our goal



Given $G \leq S_n$, describe the sequence G, $\mathrm{Comp}^{(n+1)}\,G$, $\mathrm{Comp}^{(n+2)}\,G$, $\ldots$

Our goal



Given $G \leq S_n$, describe the sequence $G, \quad \mathrm{Comp}^{(n+1)}\,G, \mathrm{Comp}^{(n+2)}\,G, \quad \ldots$

The operator $\mathrm{Comp}^{(n)}$ has a “transitive” property.

For $\ell \leq m \leq n$, $S \subseteq S_\ell$:

$$\mathrm{Comp}^{(n)}\,\mathrm{Comp}^{(m)}\,S = \mathrm{Comp}^{(n)}\,S$$

In a nutshell

Theorem

Let $G \leq S_n$ and let m be the smallest number i such that $\mathrm{Comp}^{(n+i)}\, G$ belongs to one of the stable families of groups.

1. *If G is intransitive, then $m \leq n - 1$.*
2. *If G is imprimitive and $\zeta_n \notin G$, then $m \leq p$, where p is the largest proper divisor of n.*
3. *Otherwise $m \leq 2$.*

E. Lehtonen,
Permutation groups arising from pattern involvement,
J. Algebraic Combin. **52** (2020) 251–298.