A simple proof of a CLT for vincular permutation patterns for conjugation invariant permutations

Permutation Patterns 2021

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Conjugation invariance

Definition (Conjugation invariance)

σ_n is conjugation invariant if $\rho^{-1}\sigma_n\rho \stackrel{d}{=} \sigma_n$ for any $\rho \in \mathfrak{S}_n$.

Definition (Ewens distribution)

Let $\theta \geq 0$. If $\sigma_n \sim Ew(\theta)$ then

$$\mathbb{P}(\sigma_n = \sigma) = \frac{\theta^{\#(\sigma)-1}}{\prod_{k=1}^{n-1}(\theta + k)}.$$

$\#(\sigma)$: total number of cycles of σ.

- $\theta = 1$: uniform distribution.
- $\theta = 0$: uniform distribution on permutations with a unique cycle.
- $\mathbb{E}(\#(\sigma_n)) = 1 + \sum_{k=1}^{n-1} \frac{\theta}{\theta+k} \sim \theta \log(n)$.

Vincular Patterns

A vincular pattern of size p is a couple (τ, X) such that $\tau \in \mathfrak{S}_p$ and $X \subset [p-1]$. Given $\sigma \in \mathfrak{S}_n$, an occurrence of (τ, X) is a list $i_1 < \cdots < i_p$ such that

- $i_{x+1} = i_x + 1$ for any $x \in X$.
- $(\sigma(i_1), \ldots, \sigma(i_p))$ is in the same relative order as $(\tau(i_1), \ldots, \tau(i_p))$.

We denote by $\mathcal{N}_{(\tau,X)}(\sigma)$ the number of occurrences of (τ, X) in σ.

Vincular Patterns

Theorem (Hofer (2017))

For any $\tau \in \mathfrak{S}_p$ *and any* $X \subset [p-1]$,

$$\frac{\mathcal{N}_{(\tau,X)}(\sigma_{unif,n}) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, V_{\tau,X}). \quad (1)$$

Here, $q = \mathrm{card}(X)$ *and* $V_{\tau,X} > 0$.

Féray gave a generalization for the Ewens distribution.

Main result

Proposition

Suppose that for any $n \geq 1$, σ_n *is conjugation invariant random permutation of* $\mathfrak{S}_n$ *and the sequence of number of cycles* $(\#(\sigma_n))_{n\geq 1}$ *satisfies*

$$\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0 \tag{2}$$

Then, for any $\tau \in \mathfrak{S}_p$ *and any* $X \subset [p-1]$

$$\frac{\mathcal{N}_{(\tau,X)}(\sigma_n) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, V_{\tau,X}). \tag{3}$$

Proof

Let

- $\mathrm{next}(E) :=$
  $\{\rho \circ (i,j); \rho \in E, \#(\rho \circ (i,j)) = \#(\rho) - 1\} \cup \{\rho \in E; \#(\rho) = 1\}$,
- $\mathcal{G}_{\mathfrak{S}_n}$: the directed graph with $V = \mathfrak{S}_n$ and
  $E = \{(\sigma, \rho); \sigma \in \mathfrak{S}_n, \rho \in \mathrm{next}(\{\sigma\})\}$
- T: the Markov operator associated to the uniform random walk over $\mathcal{G}_{\mathfrak{S}_n}$.

Proof

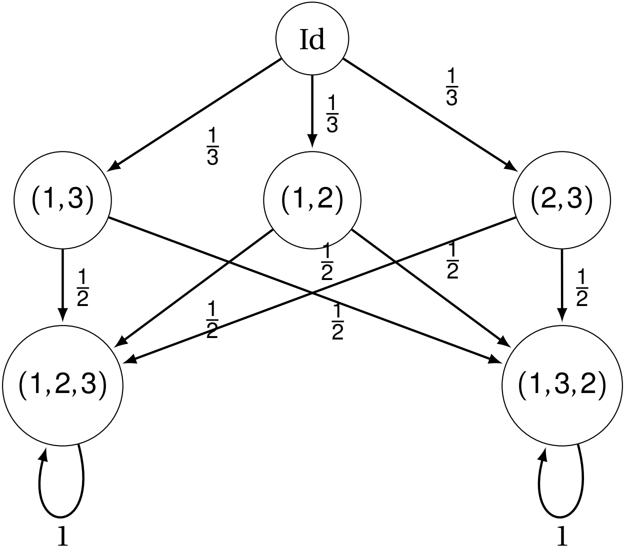


Figure: The transition probabilities of T on $\mathfrak{S}_3$

Proof

- If σ_n is conjugation invariant, then $T(\sigma_n)$ is also conjugation invariant.
- $\#(T^{n-1}(\sigma_n)) \overset{a.s}{=} \max(\#(\sigma_n) - n + 1, 1) = 1$.

Consequently, if σ_n is conjugation invariant, then:

- $T^{n-1}(\sigma_n)$ is also conjugation invariant.
- Almost surely, $\#\left(T^{n-1}(\sigma_n)\right) = 1$.

Lemma

Let σ'_n be a random permutation, if

- *σ'_n is invariant under conjugation.*
- *Almost surely, $\#(\sigma'_n) = 1$.*

Then $\sigma'_n \sim Ew(0)$.

If σ_n is conjugation invariant then $T^{n-1}(\sigma_n) \sim Ew(0)$.

Proof

Lemma

Almost surely,

$$|\mathcal{N}_{(\tau,X)}(\sigma_n) - \mathcal{N}_{(\tau,X)}(T^{n-1}(\sigma_n))| \leq n^{p-q}\#(\rho_n).$$

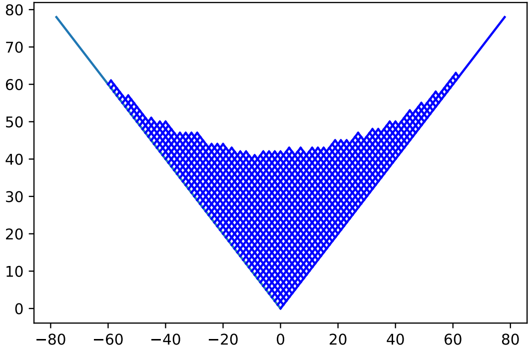
Let $f(\sigma) := \frac{\mathcal{N}_{(\tau,X)}(\sigma) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}}$. We have then $\left|f(T^{n-1}(\sigma_n)) - f(\sigma_n)\right| \leq \frac{\#(\sigma_n)}{\sqrt{n}}$

Féray proved that, $f(\sigma_{Ew,0}) \to \mathcal{N}(0, V_{\tau,X})$.

Conclusion: If $\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0$ then

$$\left|f(T^{n-1}(\sigma_n)) - f(\sigma_n)\right| \xrightarrow[n\to\infty]{d} 0.$$

If moreover σ_n is conjugation invariant then
$f(T^{n-1}(\sigma_n)) \sim f(\sigma_{Ew,0}) \to \mathcal{N}(0, V_{\tau,X})$ and then $f(\sigma_n) \to \mathcal{N}(0, V_{\tau,X})$



Thank you for your attention