Layered permutations and their density maximisers

Adam Kabela (University of West Bohemia)
joint work with Dan Král', Jon Noel and Théo Pierron

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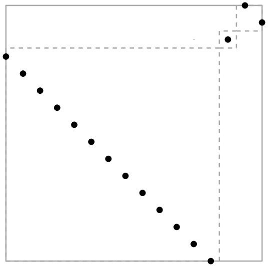
Recalling layered permutations

A permutation is *layered*

if it is a direct sum of decreasing permutations.

Equivalently a permutation is layered if it has zero density of $(3,1,2)$ and $(2,3,1)$.



For instance, the permutation $(13, 12, \ldots, 1, 14, 16, 15)$ is layered. It has three layers and their sizes are $13, 1, 2$.

Layered permutations have layered density maximisers

Theorem (Albert, Atkinson, Handley, Holton, Stromquist, 2002)

For every layered permutation π and every n, there is a layered permutation in the set of all π-maximal permutations of length n. Furthermore, if π has no singleton layer (and $n \geq |\pi|$), then every maximiser is layered.

In fact, they showed a stronger statement considening multisets of layered permutations instead of π.

Long density maximisers and the number of their layers

Proposition (for instance, Price, 1997)

Consider the $(1, 3, 2)$-maximal permutations of length n. The number of their layers goes to infinity as $n \to \infty$.

Proposition (Albert, Atkinson, Handley, Holton, Stromquist, 2002)

Let π be a permutation with no singleton layer, and consider the π-maximal permutations of length n. The number of their layers is bounded even as $n \to \infty$.

Sufficient condition for bounding the number of layers

Conjecture (Albert, Atkinson, Handley, Holton, Stromquist, 2002)

For every layered permutation π such that the first and the last layer are non-singleton and no two singleton layers are consecutive, the number of layers of π-maximal permutations of length n is bounded even as $n \to \infty$.

Disproving the conjecture of Albert et al.

Theorem (K, Král', Noel, Pierron, 2021+)

Let π be a layered permutation with layers of sizes $n, 1, \ell_1, \ldots, \ell_k$. If n is sufficiently large, then the number of layers goes to infinity for every sequence of π-maximal layered permutations of increasing lengths.

Corollary (K, Král', Noel, Pierron, 2021+)

For instance, $(13, 12, \ldots, 1, 14, 16, 15)$ is of unbounded type.

The conjecture is essentially true under an additional condition

Theorem (K, Kráľ, Noel, Pierron, 2021+)

Let π be a layered permutation whose first and last layer are non-singleton, and each pair of consecutive layers contains a non-singleton layer, and furthermore the first and last layer are of the same length and no non-singleton layer is shorter. Then every layered permuton maximising the density of π has only finitely many decreasing layers and no identity section (see the definitions in the extended abstract).

Thank you for your attention.

- M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton and W. Stromquist: On packing densities of permutations, Electron. J. Combin. 9 (2002), Research Paper 5, 20.
- A. L. Price: Packing densities of layered patterns, Ph.D. Thesis, University of Pennsylvania (1997).