On the existence of bicrucial permutations

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Joint work Carla Groenland.

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Definitions

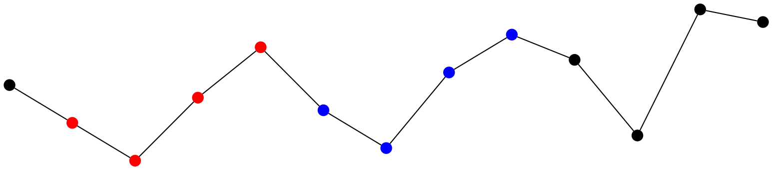
Definition

A *square* of length 2ℓ in σ is a factor

$$(S_1; S_2) = (\sigma_k, \ldots, \sigma_{k+\ell-1}; \sigma_{k+\ell}, \ldots, \sigma_{k+2\ell-1})$$

where S_1 and S_2 have the same reduced form, and we say the permutation σ is *square-free* if it contains no squares of length at least 4.

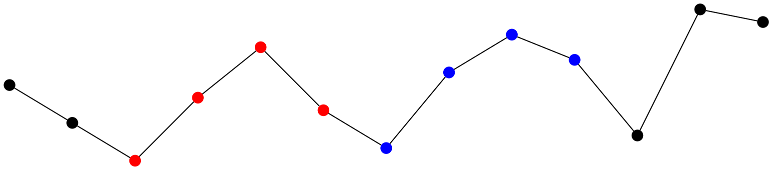
Definitions



The permutation $(7, 4, 1, 6, 10, 5, 2, 8, 11, 9, 3, 13, 12)$ is not square-free as

$$(4, 1, 6, 10) \sim (2, 1, 3, 4) \sim (5, 2, 8, 11).$$

Definitions



The factor $(1, 6, 10, 5; 2, 8, 11, 9)$ is not a square because

$$(1, 6, 10, 5) \sim (1, 3, 4, 2) \not\sim (1, 2, 4, 3) \sim (2, 8, 11, 9).$$

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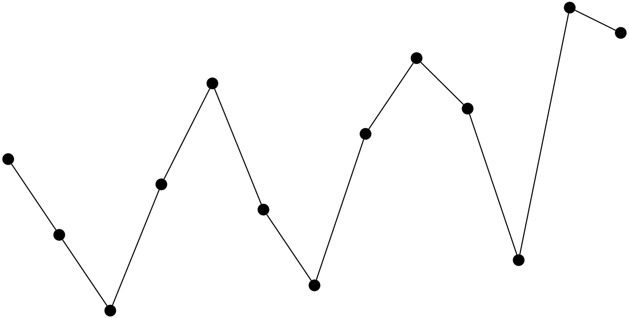

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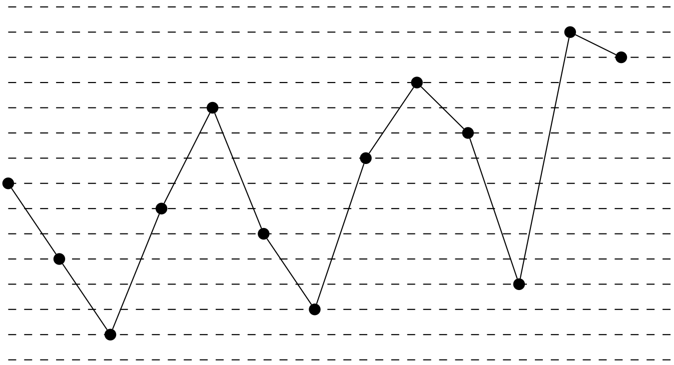

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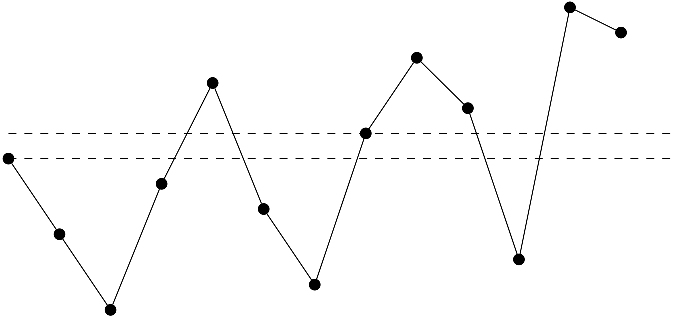

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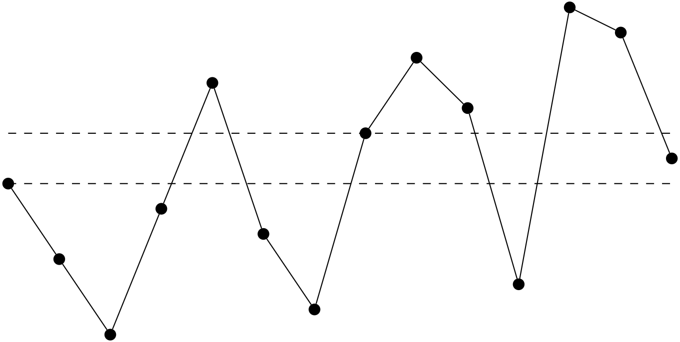

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We say that the permutation σ is *bicrucial* if it is both left-crucial and right-crucial.

Previous work

Theorem (Avgustinovich, Kitaev, Pyatkin, Valyuzhenich (2011))

There exist bicrucial permutations of length $8k+1$, $8k+5$ *and* $8k+7$ *for every* $k \geq 1$.

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Theorem (Gent, Kitaev, Konovalov, Linton, Nightingale (2015))

There exist bicrucial permutations of lengths 19, 27 *and* 32.

Conjectures

Conjecture (Gent, Kitaev, Konovalov, Linton, Nightingale (2015))

There exist bicrucial permutations of length $8k + 3$ *for all* $k \geq 2$.

Conjecture (Gent, Kitaev, Konovalov, Linton, Nightingale (2015))

There exist arbitrarily long bicrucial permutations of even length.

Bicrucial permutations

Theorem (Groenland, J. (2021+))

A bicrucial permutation of length n exists if and only if $n = 9$, $n \geq 13$ is odd or $n \geq 32$ is even and not 38.

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- Choose a left-crucial prefix and a right-crucial suffix.
- Join the prefix and suffix with a long square-free permutation which has some nice structure.
- Show that the nice structure means that there are no squares involving the prefix or the suffix.

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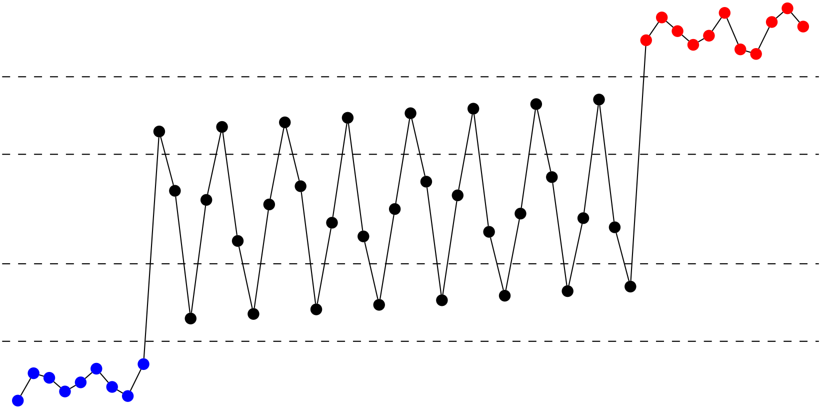
- To show there is no bicrucial permutation of length 38, use a (large) computer search.

Bicrucial permutations of length $8k + 3$



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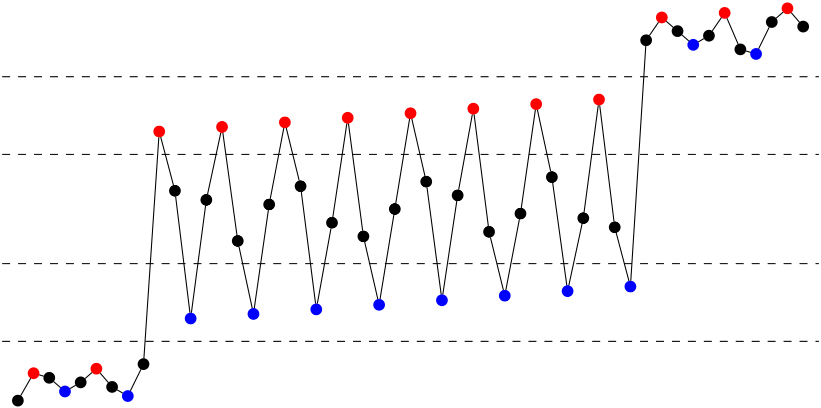
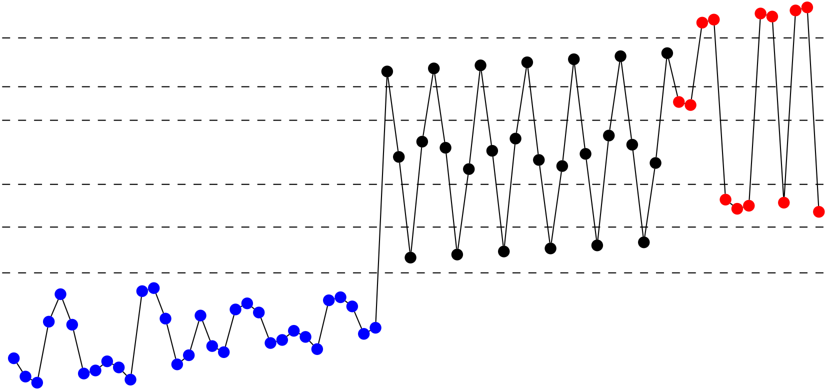


Bicrucial permutations of even length



An open problem

We say a permutation is *extremal* if it is square-free but inserting an entry in any position creates a square.

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Thanks for listening!