Sorting Time of Permutation Classes

Permutation Patterns 2021

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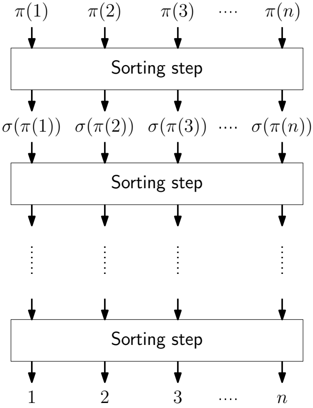
Work in progress

- This is a talk about ongoing research
- It has not been written down yet, let alone peer reviewed

Sorting



$\pi(1)$ $\pi(2)$ $\pi(3)$ $\cdots$ $\pi(n)$
Sorting step
$\sigma(\pi(1))$ $\sigma(\pi(2))$ $\sigma(\pi(3))$ $\cdots$ $\sigma(\pi(n))$
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1 2 3 $\cdots$ n

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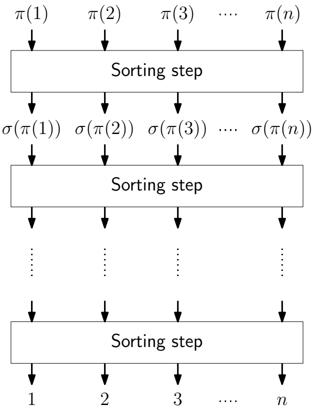

Examples of sorting steps:

- adjacent transposition (bubblesort)
- block reversal (pancake sorting, genome rearrangement)
- passage through a non-monotone stack

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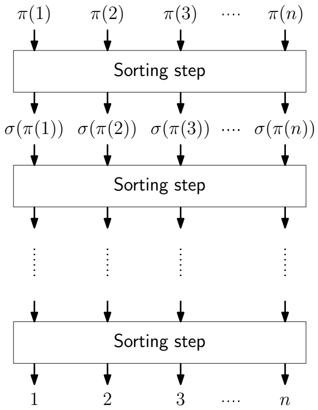

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Main question: How many steps are needed to sort any permutation of size n?

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The $\mathcal{C}$-sorting time of π, denoted $\mathrm{st}(\mathcal{C}; \pi)$, is the smallest $k \in \mathbb{N}_0$ such that π can be mapped to the identity permutation by a composition of k sorting steps. If no such k exists, we put $\mathrm{st}(\mathcal{C}; \pi) = +\infty$.

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The worst-case $\mathcal{C}$-sorting time, denoted $\mathrm{wst}(\mathcal{C}; n)$, is defined as

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Main result

For any permutation class $\mathcal{C}$, one of the following holds:

1. $\mathrm{wst}(\mathcal{C}; n) = 1$ *for all* $n \in \mathbb{N}$,
2. $\Omega(\log n) \leq \mathrm{wst}(\mathcal{C}; n) \leq O(\log^2 n)$,
3. $\Omega(\sqrt{n}) \leq \mathrm{wst}(\mathcal{C}; n) \leq O(n)$,
4. $\mathrm{wst}(\mathcal{C}; n) = \Theta(n^2)$, *or*
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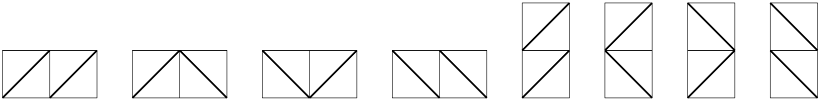
In all cases where we could determine the asymptotics of $\mathrm{wst}(\mathcal{C}; n)$, we found either 1, $\Theta(\log n)$, $\Theta(n)$, $\Theta(n^2)$, or $+\infty$.

Classes with polylog sorting time

A monotone juxtaposition is any of the following eight classes:



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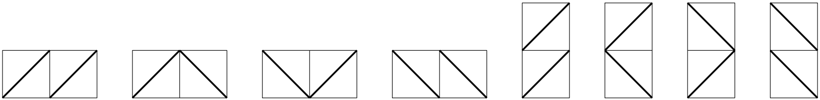

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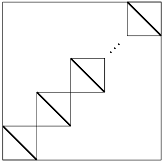
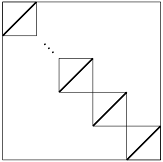
For $\mathcal{C}$ a monotone juxtaposition, $\mathrm{wst}(\mathcal{C}; n) = \Theta(\log n)$.

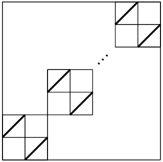
Consider the following four classes:



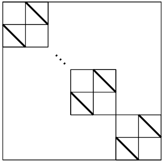
layered ($\mathcal{L}$)



co-layered ($\mathcal{L}^c$)



$\mathcal{PBT}$



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For any $\mathcal{C} \in \{\mathcal{L}, \mathcal{L}^c, \mathcal{PBT}, \mathcal{PBT}^c\}$, we have $\Omega(\log n) \leq \mathrm{wst}(\mathcal{C}; n) \leq O(\log^2 n)$.

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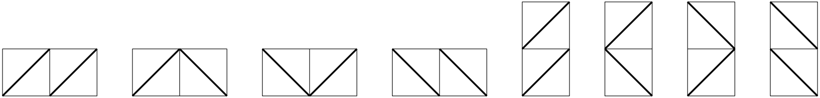

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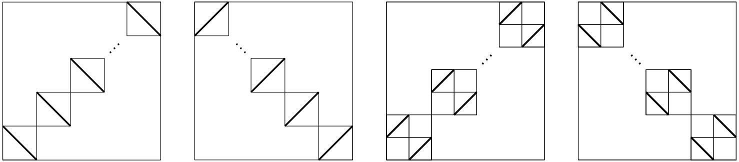

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Open problem: Close the gap.

From polylog to polynomial

Theorem

If $\mathcal{C}$ does not contain any monotone juxtaposition and any of $\mathcal{L}$, $\mathcal{L}^c$, $\mathcal{PBT}$, and $\mathcal{PBT}^c$ as subclass, then $\mathrm{wst}(\mathcal{C}; n) \geq \Omega(\sqrt{n})$.

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Conjecture: The lower bound can be improved to $\Omega(n)$.

The End

Thank you for your attention!